

Generalized quantum Chernoff bound

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Abstract—We consider the task of distinguishing whether a quantum system is prepared in a state belonging to one of several sets of quantum states. Assuming convexity and stability under tensor product, we prove that the optimal error exponent for discrimination is precisely given by the regularized quantum Chernoff divergence between the sets, thereby establishing a generalized Chernoff bound for the discrimination of multiple sets of quantum states. This extends the classical and quantum Chernoff bounds to the general setting of composite and correlated quantum hypotheses. Furthermore, leveraging minimax theorems, we show that discriminating between sets of quantum states is no harder than discriminating between their worst-case elements in terms of error probability. This implies the existence of an optimal state-agnostic test that achieves the minimum error probability for all states in the sets, matching the performance of the optimal state-dependent test for the most difficult pair of states. We provide explicit characterizations of the optimal state-agnostic test in the binary composite case. Finally, we show that the maximum overlap between a pure state and a set of free states, a quantity that frequently arises in quantum resource theories, equals the quantum Chernoff divergence between the state and the set, thereby providing an operational interpretation of this quantity in the context of symmetric hypothesis testing.

Index Terms—Quantum hypothesis testing, Quantum Chernoff bound, Composite hypotheses, Correlated hypotheses

I. INTRODUCTION

Hypothesis testing is a fundamental method for deciding between competing explanations of observed data. It provides a rigorous framework for making decisions under uncertainty and is central to statistics, information theory, and many applied fields such as signal processing, machine learning,

and experimental physics [1]. The main goal of hypothesis testing is to determine which of several possible models or probability distributions best describes a set of observations, while keeping the probability of making an error as low as possible. In the classical setting, Chernoff showed that, when many independent samples are available, the probability of error decreases exponentially with the number of samples. The rate at which this error probability decays is precisely given by the celebrated Chernoff bound [2], which quantifies the fundamental limit of distinguishability between two hypotheses.

The Chernoff bound has been extended to the quantum setting, where the task is to distinguish between two quantum hypotheses: the system is prepared either in state ρ_1 (the null hypothesis) or in state ρ_2 (the alternative hypothesis). In the Bayesian framework, each hypothesis is assigned a prior probability, denoted by π_1 and π_2 . Operationally, discrimination is performed using a two-outcome positive operator-valued measure (POVM) $\{M_1, M_2\}$, where $M_1 + M_2 = I$. According to Born's rule in quantum mechanics, the average error probability for this measurement is given by $P_e(\{M_1, M_2\}, \{\rho_1, \rho_2\}) := \pi_1 \text{Tr}[\rho_1(I - M_1)] + \pi_2 \text{Tr}[\rho_2(I - M_2)]$. The objective is to minimize the average error probability over all possible POVMs: $P_{e,\min}(\{\rho_1, \rho_2\}) := \inf \{P_e(\{M_1, M_2\}, \{\rho_1, \rho_2\}) : \{M_1, M_2\} \in \text{POVM}\}$.

A landmark result, known as the quantum Chernoff bound, was established by Audenaert et al. [3] and Nussbaum and Szkoła [4], who showed that, in the asymptotic regime, the optimal error probability decays exponentially with the number of copies, with the optimal exponent given by

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log P_{e,\min}(\{\rho_1^{\otimes n}, \rho_2^{\otimes n}\}) = C(\rho_1 \| \rho_2),$$

where $C(\rho_1 \| \rho_2) := \max_{0 \leq \alpha \leq 1} -\log Q_\alpha(\rho_1 \| \rho_2)$ is the Chernoff divergence with $Q_\alpha(\rho_1 \| \rho_2) := \text{Tr}[\rho_1^\alpha \rho_2^{1-\alpha}]$.

The quantum Chernoff bound has also been extended to the discrimination of multiple quantum hypotheses. Consider r quantum states $\{\rho_i\}_{i=1}^r$ with prior probabilities $\{\pi_i\}_{i=1}^r$. Let $\{M_i\}_{i=1}^r$ be a POVM,

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i.e., a collection of positive semidefinite operators that sum to the identity. The average error probability for discriminating among these r quantum states is given by $P_e(\{M_i\}_{i=1}^r, \{\rho_i\}_{i=1}^r) := \sum_{i=1}^r \pi_i \text{Tr}[\rho_i(I - M_i)]$. Optimizing over all POVMs, the minimum error probability is defined as $P_{e,\min}(\{\rho_i\}_{i=1}^r) := \inf \{P_e(\{M_i\}_{i=1}^r, \{\rho_i\}_{i=1}^r) : \{M_i\}_{i=1}^r \in \text{POVM}\}$. Li showed [5] that, in the asymptotic regime,

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log P_{e,\min}(\{\rho_i^{\otimes n}\}_{i=1}^r) = \min_{i \neq j} C(\rho_i \| \rho_j). \quad (1)$$

This result generalizes the binary quantum Chernoff bound to the case of multiple quantum states, showing that the optimal error exponent is determined by the pair of states that are hardest to distinguish.

A. Quantum Chernoff bound of multiple sets of states

In many practical scenarios, the states to be distinguished are not fully specified (i.e., *composite hypotheses* [6]–[12]), such as in adversarial or black-box settings [13], [14], and may exhibit correlations that preclude a simple tensor product structure (i.e., *correlated hypotheses* [15]–[18]). In this setting, we aim to distinguish between several *sets of quantum states*, where each set $\mathcal{C}_{i,n}$ may contain correlated states. A tester receives a quantum state drawn from one of these sets and must decide, using a measurement $\{M_{i,n}\}_{i=1}^r$, which set the state came from. Formally, let $\mathcal{C}_{i,n}$ for $i = 1, \dots, r$ be r sets of quantum states, each with prior probability π_i (where $\sum_{i=1}^r \pi_i = 1$). The measurement $\{M_{i,n}\}_{i=1}^r$ is a POVM, with $M_{i,n}$ corresponding to the decision that the state is from $\mathcal{C}_{i,n}$. For any choice of states $\rho_{i,n} \in \mathcal{C}_{i,n}$, the probability of error is $\sum_{i=1}^r \pi_i \text{Tr}[\rho_{i,n}(I - M_{i,n})]$. Since the actual states are unknown, we consider the worst-case error over all possible choices: $P_e(\{M_{i,n}\}_{i=1}^r, \{\mathcal{C}_{i,n}\}_{i=1}^r) := \sup_{\rho_{i,n} \in \mathcal{C}_{i,n}} \sum_{i=1}^r \pi_i \text{Tr}[\rho_{i,n}(I - M_{i,n})]$. To characterize the fundamental limit of discrimination, we minimize the worst-case error probability over all possible POVMs: $P_{e,\min}(\{\mathcal{C}_{i,n}\}_{i=1}^r) := \inf \{P_e(\{M_{i,n}\}_{i=1}^r, \{\mathcal{C}_{i,n}\}_{i=1}^r) : \{M_{i,n}\}_{i=1}^r \in \text{POVM}\}$. We are interested in the asymptotic behavior of the minimum error probability:

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log P_{e,\min}(\{\mathcal{C}_{i,n}\}_{i=1}^r) = ? \quad (2)$$

B. Challenges for composite correlated hypotheses

Discriminating between composite and correlated hypotheses (i.e., sets of quantum states) presents several significant challenges. First, discrimination strategies must be state-agnostic, ensuring that the error probability is universally controlled for every possible state within each set, regardless of which specific state is sampled. Second, the optimization problem naturally takes a minimax form, requiring a simultaneous maximization over all possible states in the sets and minimization over all quantum measurements. Third, the absence of an i.i.d. structure complicates the asymptotic analysis, as standard techniques based on tensor product

structures are no longer directly applicable. Fourth, it becomes necessary to define suitable extensions of the quantum Chernoff divergence to sets of quantum states that both recover known results for i.i.d. sources and accurately capture the essential features of the general composite correlated setting.

Existing results do not directly address this general scenario. While recent progress has been made in the asymmetric (Stein's) regime for binary hypotheses, where the type-II error is minimized subject to a constraint on the type-I error [19], the symmetric hypothesis testing for multiple sets of quantum states remains open. In this work, we resolve this gap by establishing generalized quantum Chernoff bounds for the discrimination of multiple sets of quantum states, thereby extending the quantum Chernoff bound to the general composite and correlated setting.

C. Summary of main results

(Chernoff bounds:) We establish generalized quantum Chernoff bounds for the discrimination of multiple sets of quantum states. Theorem 2 demonstrates that, under the assumptions of convexity and stability under tensor product, the optimal error exponent in Eq. (2) is exactly characterized by the regularized Chernoff divergence between the sets. These results unify and extend the classical and quantum Chernoff bounds to the general setting of composite and correlated hypotheses, encompassing several previously known results as special cases.

(Optimal tests:) In Theorem 3, we show that discriminating between sets of quantum states is no harder than discriminating between their most difficult elements, in terms of error probability. This minimax result guarantees the existence of an optimal state-agnostic test that achieves the minimum error probability for all states in the sets, matching the performance of the optimal state-dependent test for the hardest pair of states. Furthermore, Theorem 4 provides an explicit construction of the optimal state-agnostic test via the Holevo-Helstrom test in the binary composite hypothesis setting.

(Operational interpretation:) Theorem 5 provides an operational interpretation of the maximum overlap between a pure state and a set of free states, a quantity that frequently arises in quantum resource theories, as the optimal error exponent in symmetric hypothesis testing, justifying the overlap as a meaningful measure of distinguishability in this context.

D. Notation

We adopt the following notational conventions throughout this work. Finite-dimensional Hilbert spaces are denoted by $\mathcal{H}$, with $|\mathcal{H}|$ representing their dimension. The set of all linear operators on $\mathcal{H}$ is denoted by $\mathcal{L}(\mathcal{H})$, while $\mathcal{H}(\mathcal{H})$ and $\mathcal{H}_+(\mathcal{H})$ denote the sets of Hermitian and positive semidefinite operators on $\mathcal{H}$, respectively. The set of density matrices (i.e., positive semidefinite operators with unit trace) on $\mathcal{H}$ is denoted by $\mathcal{D}(\mathcal{H})$. Calligraphic letters such as $\mathcal{A}$, $\mathcal{B}$,

and $\mathcal{C}$ are used to denote sets or sequences of sets of linear operators. Unless otherwise specified, all logarithms are taken to base two and denoted by $\log(x)$. The positive semidefinite ordering is written as $X \geq Y$ if and only if $X - Y \geq 0$. The absolute value of an operator X is defined as $|X| := (X^\dagger X)^{1/2}$. For a Hermitian operator X with spectral decomposition $X = \sum_i x_i E_i$, the projection onto the non-negative eigenspaces is denoted by $\{X \geq 0\} := \sum_{x_i \geq 0} E_i$. The Petz Rényi divergence is defined by $D_{P,\alpha}(\rho_1 \parallel \rho_2) := \frac{1}{\alpha-1} \log Q_\alpha(\rho_1 \parallel \rho_2)$ and its extension to two sets of quantum states $\mathcal{C}_1$ and $\mathcal{C}_2$ is defined as $D_{P,\alpha}(\mathcal{C}_1 \parallel \mathcal{C}_2) := \inf_{\rho_1 \in \mathcal{C}_1, \rho_2 \in \mathcal{C}_2} D_{P,\alpha}(\rho_1 \parallel \rho_2)$.

II. QUANTUM CHERNOFF DIVERGENCE

In this section, we introduce the quantum Chernoff divergence between sets of states, which generalizes the concept from individual states to sets and serves as a fundamental measure of their distinguishability. For convex sets, this divergence admits a minimax characterization as shown in Theorem 1 in terms of the Chernoff quasi-divergence. This relation extends to the asymptotic case for stable sequences of sets.

Definition 1 (Quantum Chernoff divergence between two sets of quantum states). *Let $\mathcal{H}$ be a finite-dimensional Hilbert space, and let $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{D}(\mathcal{H})$ be two sets of quantum states. The quantum Chernoff divergence between sets is defined as*

$$C(\mathcal{C}_1 \parallel \mathcal{C}_2) := \inf \{C(\rho_1 \parallel \rho_2) : \rho_1 \in \mathcal{C}_1, \rho_2 \in \mathcal{C}_2\}.$$

Moreover, let $\mathcal{C}_1 = \{\mathcal{C}_{1,n}\}_{n \in \mathbb{N}}$ and $\mathcal{C}_2 = \{\mathcal{C}_{2,n}\}_{n \in \mathbb{N}}$ be two sequences of sets of quantum states, where each $\mathcal{C}_{1,n}, \mathcal{C}_{2,n} \subseteq \mathcal{D}(\mathcal{H}^{\otimes n})$. The regularized quantum Chernoff divergence between these sequences is defined as

$$\begin{aligned} \underline{C}^\infty(\mathcal{C}_1 \parallel \mathcal{C}_2) &:= \liminf_{n \rightarrow \infty} \frac{1}{n} C(\mathcal{C}_{1,n} \parallel \mathcal{C}_{2,n}), \\ \overline{C}^\infty(\mathcal{C}_1 \parallel \mathcal{C}_2) &:= \limsup_{n \rightarrow \infty} \frac{1}{n} C(\mathcal{C}_{1,n} \parallel \mathcal{C}_{2,n}). \end{aligned}$$

If the following limit exists, we define

$$C^\infty(\mathcal{C}_1 \parallel \mathcal{C}_2) := \lim_{n \rightarrow \infty} \frac{1}{n} C(\mathcal{C}_{1,n} \parallel \mathcal{C}_{2,n}).$$

In many applications, the sequences under consideration are not arbitrary but possess a structure that is compatible with tensor products. This property, known as *stability* (or *closeness*) under tensor product, is formalized as follows.

Definition 2 (Stable sequence). *Let $\mathcal{C}_1 \subseteq \mathcal{H}_+(\mathcal{H}_1)$, $\mathcal{C}_2 \subseteq \mathcal{H}_+(\mathcal{H}_2)$, and $\mathcal{C}_3 \subseteq \mathcal{H}_+(\mathcal{H}_1 \otimes \mathcal{H}_2)$. We say that $\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$ is stable under tensor product if, for any $X_1 \in \mathcal{C}_1$ and $X_2 \in \mathcal{C}_2$, it holds that $X_1 \otimes X_2 \in \mathcal{C}_3$. In short, we write $\mathcal{C}_1 \otimes \mathcal{C}_2 \subseteq \mathcal{C}_3$. A sequence of sets $\{\mathcal{C}_n\}_{n \in \mathbb{N}}$ with $\mathcal{C}_n \subseteq \mathcal{H}_+(\mathcal{H}^{\otimes n})$ is stable if $\mathcal{C}_n \otimes \mathcal{C}_m \subseteq \mathcal{C}_{n+m}$ for all $n, m \in \mathbb{N}$.*

Remark 1. *As a consequence of the subadditivity of the Chernoff divergence, its extension to sets of quantum states is also subadditive, provided that the sequences $\mathcal{C}_1 =$*

$\{\mathcal{C}_{1,n}\}_{n \in \mathbb{N}}$ and $\mathcal{C}_2 = \{\mathcal{C}_{2,n}\}_{n \in \mathbb{N}}$ are both stable under tensor product [19, Lemma 26]. Therefore, the regularized quantum Chernoff divergence exists by Fekete's lemma and satisfies $C^\infty(\mathcal{C}_1 \parallel \mathcal{C}_2) = \overline{C}^\infty(\mathcal{C}_1 \parallel \mathcal{C}_2) = \underline{C}^\infty(\mathcal{C}_1 \parallel \mathcal{C}_2) = \inf_{n \geq 1} \frac{1}{n} C(\mathcal{C}_{1,n} \parallel \mathcal{C}_{2,n})$.

Analogous to the extension of the quantum Chernoff divergence to sets of quantum states, we also generalize the quantum Chernoff quasi-divergence.

Definition 3 (Quantum Chernoff quasi-divergence between two sets of quantum states). *Let $\alpha \in [0, 1]$ and $\mathcal{H}$ be a finite-dimensional Hilbert space. Let $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{D}(\mathcal{H})$ be two sets of quantum states. The quantum Chernoff quasi-divergence between these sets is defined as*

$$Q_\alpha(\mathcal{C}_1 \parallel \mathcal{C}_2) := \sup \{Q_\alpha(\rho_1 \parallel \rho_2) : \rho_1 \in \mathcal{C}_1, \rho_2 \in \mathcal{C}_2\}.$$

Moreover, let $\mathcal{C}_1 = \{\mathcal{C}_{1,n}\}_{n \in \mathbb{N}}$ and $\mathcal{C}_2 = \{\mathcal{C}_{2,n}\}_{n \in \mathbb{N}}$ be two sequences of sets of quantum states, where each $\mathcal{C}_{1,n}, \mathcal{C}_{2,n} \subseteq \mathcal{D}(\mathcal{H}^{\otimes n})$. The regularized quantum Chernoff quasi-divergence between these sequences is defined as

$$Q_\alpha^\infty(\mathcal{C}_1 \parallel \mathcal{C}_2) := \lim_{n \rightarrow \infty} [Q_\alpha(\mathcal{C}_{1,n} \parallel \mathcal{C}_{2,n})]^{1/n},$$

provided the limit exists.

While the quantum Chernoff divergence C and quasi-divergence Q_α between two states are directly related through their definitions, we have thus far extended these quantities for sets of quantum states independently. It is therefore natural to ask whether a relationship analogous to $C(\rho_1 \parallel \rho_2) = \max_{0 \leq \alpha \leq 1} -\log Q_\alpha(\rho_1 \parallel \rho_2)$ holds between the Chernoff divergence and quasi-divergence when extended to sets of quantum states. The following result confirms this connection by applying minimax theorems for convex sets.

Theorem 1 (Finite and asymptotic connections). *Let $\mathcal{H}$ be a finite-dimensional Hilbert space, and let $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{D}(\mathcal{H})$ be two convex sets of quantum states. Then it holds that*

$$C(\mathcal{C}_1 \parallel \mathcal{C}_2) = \max_{\alpha \in [0,1]} -\log Q_\alpha(\mathcal{C}_1 \parallel \mathcal{C}_2).$$

Moreover, let $\mathcal{C}_1 = \{\mathcal{C}_{1,n}\}_{n \in \mathbb{N}}$ and $\mathcal{C}_2 = \{\mathcal{C}_{2,n}\}_{n \in \mathbb{N}}$ be two stable sequences of convex sets of quantum states, where each $\mathcal{C}_{1,n}, \mathcal{C}_{2,n} \subseteq \mathcal{D}(\mathcal{H}^{\otimes n})$. Then it holds that

$$C^\infty(\mathcal{C}_1 \parallel \mathcal{C}_2) = \max_{\alpha \in [0,1]} -\log Q_\alpha^\infty(\mathcal{C}_1 \parallel \mathcal{C}_2).$$

III. QUANTUM CHERNOFF BOUND FOR MULTIPLE SETS OF QUANTUM STATES

In this section, we establish the quantum Chernoff bound for the discrimination of multiple sets of quantum states. We provide both lower and upper bounds on the asymptotic error exponent and show that, under suitable structural assumptions, the bounds coincide and the optimal exponent is given by the regularized Chernoff divergence between the sets.

Assumption 1. *We use the following assumptions for a sequence of sets of quantum states $\mathcal{C} = \{\mathcal{C}_n\}_{n \in \mathbb{N}}$, where*

each $\mathcal{C}_n \subseteq \mathcal{D}(\mathcal{H}^{\otimes n})$. (C1) Convexity: For any $n \in \mathbb{N}$, the set $\mathcal{C}_n$ is convex; (C2) Stability under tensor product: For any $m, n \in \mathbb{N}$, it holds that $\mathcal{C}_m \otimes \mathcal{C}_n \subseteq \mathcal{C}_{m+n}$; (C3) Finiteness: $D_{P, \alpha}(\mathcal{C}_1 \| \mathcal{C}'_1) < \infty$ for $\alpha \in (0, 1)$ if considering two sequences $\mathcal{C}, \mathcal{C}'$.¹

Analogous to the result for multiple quantum states in Eq. (1), we demonstrate that the optimal error exponent is characterized by the smallest regularized Chernoff divergence among all pairs of sets.

Theorem 2 (Quantum Chernoff bound for multiple hypotheses). *Let $\mathcal{H}$ be a finite-dimensional Hilbert space, and let $\mathcal{C}_i = \{\mathcal{C}_{i,n}\}_{n \in \mathbb{N}}$ for $i \in \{1, \dots, r\}$ be r sequences of sets of quantum states, where each $\mathcal{C}_{i,n} \subseteq \mathcal{D}(\mathcal{H}^{\otimes n})$. Let $\{\pi_i\}_{i=1}^r$ be the prior probabilities.*

- If the sequences $\{\mathcal{C}_i\}_{i=1}^r$ satisfy assumption (C1):

$$\liminf_{n \rightarrow \infty} -\frac{1}{n} \log P_{e, \min}(\{\mathcal{C}_{i,n}\}_{i=1}^r) \geq \min_{i \neq j} C^\infty(\mathcal{C}_i \| \mathcal{C}_j).$$

- If the sequences $\{\mathcal{C}_i\}_{i=1}^r$ satisfy assumptions (C2), (C3):

$$\limsup_{n \rightarrow \infty} -\frac{1}{n} \log P_{e, \min}(\{\mathcal{C}_{i,n}\}_{i=1}^r) \leq \min_{i \neq j} C^\infty(\mathcal{C}_i \| \mathcal{C}_j).$$

Consequently, if the sequences $\{\mathcal{C}_i\}_{i=1}^r$ satisfy assumptions (C1), (C2) and (C3), then the following limit exists and

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log P_{e, \min}(\{\mathcal{C}_{i,n}\}_{i=1}^r) = \min_{i \neq j} C^\infty(\mathcal{C}_i \| \mathcal{C}_j).$$

Remark 2. If we choose $\mathcal{C}_{i,n} = \{\rho_i^{\otimes n}\}$ as the singleton i.i.d. quantum states, then we can recover the quantum Chernoff bound for multiple quantum states in Eq. (1). Note that for general sets $\mathcal{C}_{i,n}$, i.e., not necessarily convex, we have $P_{e, \min}(\{\mathcal{C}_i\}_{i=1}^r) = P_{e, \min}(\{\text{conv}(\mathcal{C}_i)\}_{i=1}^r)$ where $\text{conv}(\mathcal{C})$ denotes the convex hull of set $\mathcal{C}$. Therefore, we have $\lim_{n \rightarrow \infty} -\frac{1}{n} \log P_{e, \min}(\{\mathcal{C}_{i,n}\}_{i=1}^r) = \min_{i \neq j} C^\infty(\text{conv}(\mathcal{C}_i) \| \text{conv}(\mathcal{C}_j))$ where $\text{conv}(\mathcal{C}_i) = \{\text{conv}(\mathcal{C}_{i,n})\}_{n \in \mathbb{N}}$ denotes the sequence of convex hulls. In the case $r = 2$, our result also strengthens the result of Mosonyi et al. [9, Eq. (II.62)] by establishing that the bound therein actually holds with equality.

Compared with recent advances in asymmetric hypothesis testing between two sets of quantum states (i.e., generalized quantum Stein's lemmas) [10], [19], [20], the quantum Chernoff bound for multiple sets of quantum states established in Theorem 2 requires substantially weaker assumptions. In particular, it does not rely on permutation invariance or full-rank assumptions as in [10], [20], nor on the polar assumption used in [19], which are essential in those works for their proofs. It is moreover known that, under assumptions (C1)–(C3) alone, there exist counterexamples where the generalized quantum Stein's lemma fails [21], [22]. This highlights both the generality of our result and a fundamental distinction

¹This is a mild technical assumption, requiring that there exist $\rho \in \mathcal{C}_1$ and $\rho' \in \mathcal{C}'_1$ that are not orthogonal.

between symmetric and asymmetric hypothesis testing in the setting of composite and correlated quantum hypotheses.

The proof of this result is highly nontrivial and extends far beyond the techniques used in existing quantum Chernoff bounds. As highlighted in Section I-B, numerous challenges arise when considering composite correlated hypotheses. Since the relevant sequences of states are no longer i.i.d., their analysis is more involved and requires the use of Nussbaum-Szkoła distributions [4] and the Gärtner-Ellis theorem [23]. We also introduce new techniques for constructing optimal sequences of states in the achievability part of the proof, as well as a new discretization method to control the spectra of the states within the sets, which allows us to remove the reliance on permutation invariance and full-rank assumptions on the sequences. The complete proof is provided in [24].

IV. OPTIMAL ERROR PROBABILITY AND OPTIMAL TEST FOR SETS OF STATES

In this section, we present a minimax formula for the optimal average error probability when distinguishing between multiple sets of quantum states. Specifically, the minimum error probability for composite hypotheses equals the maximum, over all possible choices of states from each set, of the minimum error probability for those states. This reduces the composite problem to the hardest pairwise discrimination. For binary hypotheses, we also give an explicit construction of the optimal state-agnostic test, which is the Holevo-Helstrom test for the most difficult pair of states.

A. Optimal average error probability

The following establishes a fundamental relation between discriminating sets of quantum states and discriminating their most challenging elements, based on the minimax theorem.

Theorem 3 (Optimal error probability). *Let $\mathcal{H}$ be a finite-dimensional Hilbert space, and let $\{\mathcal{C}_i\}_{i=1}^r$ be r convex sets of quantum states, where each $\mathcal{C}_i \subseteq \mathcal{D}(\mathcal{H})$. Then*

$$P_{e, \min}(\{\mathcal{C}_i\}_{i=1}^r) = \sup_{\forall i, \rho_i \in \mathcal{C}_i} P_{e, \min}(\{\rho_i\}_{i=1}^r),$$

where the suprema can be replaced by maxima if the sets are also compact.

B. Optimal test for binary composite hypotheses

In quantum hypothesis testing between two quantum states ρ_1 and ρ_2 with prior probabilities π_1 and π_2 , the optimal measurement is given by the Holevo-Helstrom test $\{\pi_1 \rho_1 - \pi_2 \rho_2 \geq 0\}$ [25], [26]. However, for composite hypotheses, where only partial information about the states is available, the problem becomes much more challenging. Here, one must design a test that *universally* minimizes the average error

probability for all possible states within the specified sets. In particular, the minimax equality

$$\inf_{\{M_i\}_{i=1}^r} \sup_{\forall i, \rho_i \in \mathcal{C}_i} \sum_{i=1}^r \pi_i \text{Tr}[\rho_i(I - M_i)] \\ = \sup_{\forall i, \rho_i \in \mathcal{C}_i} \inf_{\{M_i\}_{i=1}^r} \sum_{i=1}^r \pi_i \text{Tr}[\rho_i(I - M_i)], \quad (3)$$

guarantees the existence of such an optimal state-agnostic test: one that achieves the minimum error probability for all states in the sets (i.e., the minimizer on the left-hand side), matching the performance of the optimal state-dependent test for the most difficult pair of states (i.e., the minimizer on the right-hand side). While the minimax theorem ensures the existence of an optimal state-agnostic test, it does not provide an explicit construction.

Here, we explicitly construct an optimal test for binary composite hypotheses by analyzing the saddle point (or equilibrium point) of the minimax problem in more detail.

1) *Minimax optimization and saddle points:* To obtain saddle points, we may calculate the inner “sup” and “inf” functions appearing in Eq. (3), then minimize and maximize them, respectively. While this standard approach to finding saddle points involves evaluating both sides of the minimax inequality, we provide an alternative method below. It allows for the construction of saddle points by optimizing only one side of the minimax problem, which will prove useful in the explicit construction of optimal tests later.

Lemma 1 (Saddle point by one-side optimization). *Let $f : X \times Y \rightarrow \mathbb{R}$ and $g(y) = \inf_{x \in X} f(x, y)$. Suppose $y^* \in Y$ is a maximizer of $g(y)$, i.e., $g(y^*) = \sup_{y \in Y} g(y)$, and $x^* \in X$ is a minimizer of $\inf_{x \in X} f(x, y^*)$, i.e., $f(x^*, y^*) = \inf_{x \in X} f(x, y^*)$. If the minimax equality holds for function f and the optimization $\inf_{x \in X} f(x, y^*)$ has a unique minimizer, then (x^*, y^*) is a saddle point. Consequently, x^* is a minimizer of the optimization $\inf_{x \in X} [\sup_{y \in Y} f(x, y)]$.*

We note that the uniqueness of the minimizer in $\inf_{x \in X} f(x, y^*)$ is crucial for Lemma 1 to apply. If the minimizer is not unique, it is generally unclear whether any minimizer will yield a saddle point. In particular, not every minimizer of $\inf_{x \in X} f(x, y^*)$ is necessarily a minimizer of $\inf_{x \in X} [\sup_{y \in Y} f(x, y)]$. This is illustrated by the following counterexample: $\min_{x \in [-1, 1]} \max_{y \in [-1, 1]} xy = \max_{y \in [-1, 1]} \min_{x \in [-1, 1]} xy$. If we solve $\max_{y \in [-1, 1]} (\min_{x \in [-1, 1]} xy)$, we find $y^* = 0$, and any $x^* \in [-1, 1]$ is a minimizer of $\min_{x \in [-1, 1]} xy^*$. However, only $x^* = 0$ is a minimizer of $\min_{x \in [-1, 1]} (\max_{y \in [-1, 1]} xy)$; that is, only $(0, 0)$ is a saddle point.

Lemma 2. *Let $X \in \mathcal{H}$ be a full rank Hermitian operator. Then the optimal solution to the semidefinite program $\max_{0 \leq M \leq I} \text{Tr}[XM]$ is unique and is given by $\{X \geq 0\}$.*

2) *Optimal test for binary hypotheses:* With Lemma 1 and Lemma 2 in place, we are now prepared to explicitly construct the optimal test. The following result shows that, for binary composite hypotheses, the optimal state-agnostic measurement is given by the Holevo-Helstrom test corresponding to the pair of states from the sets that achieve the minimal trace distance.

Theorem 4 (Optimal test for binary hypotheses). *Let $\mathcal{H}$ be a finite-dimensional Hilbert space. Let $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{D}(\mathcal{H})$ be two convex, compact sets of quantum states and $\{\pi_1, \pi_2\}$ be the prior probabilities. Suppose $(\rho_1^*, \rho_2^*) \in \mathcal{C}_1 \times \mathcal{C}_2$ is a minimizer of the convex optimization problem $\min_{\rho_1 \in \mathcal{C}_1, \rho_2 \in \mathcal{C}_2} \|\pi_1 \rho_1 - \pi_2 \rho_2\|_1$. Then, the optimal state-agnostic test for discriminating between $\mathcal{C}_1$ and $\mathcal{C}_2$ is given by the projection $\{\pi_1 \rho_1^* - \pi_2 \rho_2^* \geq 0\}$, provided that $\pi_1 \rho_1^* - \pi_2 \rho_2^*$ is full rank.*

V. MAXIMUM OVERLAP IN RESOURCE THEORY

The maximum overlap between a pure state $|\psi\rangle$ and a set of free states $\mathcal{F}$, $O_{\mathcal{F}}(\psi) := \sup_{\sigma \in \mathcal{F}} \langle \psi | \sigma | \psi \rangle$, is a technical quantity that appears frequently in quantum resource theory [27]–[29]. Here, we provide an operational interpretation of this quantity as the optimal error exponent in symmetric hypothesis testing.

Theorem 5. *Let $|\psi\rangle\langle\psi|$ be a pure state, and let $\mathcal{F} \subseteq \mathcal{D}$ be a convex set of quantum states. Then we have the relation $C(|\psi\rangle\langle\psi| \| \mathcal{F}) = -\log O_{\mathcal{F}}(\psi)$.*

This connection justifies the maximum overlap as a meaningful resource quantifier. Furthermore, it is known from [30, Section 6.2] that there exist quantum states for which the maximum overlap with stabilizer states is *not multiplicative*. Consequently, the quantum Chernoff divergence between two sets is not additive in general, thereby justifying again the necessity of regularization for the quantum Chernoff divergence.

VI. DISCUSSION

We have established generalized quantum Chernoff bounds for the discrimination of multiple sets of quantum states, thereby extending the classical and quantum Chernoff bounds to the general setting of composite and correlated quantum hypotheses. Our main results show that the optimal asymptotic error exponent for discriminating between stable sequences of convex sets of quantum states is exactly given by the regularized Chernoff divergence between the sets. The minimal assumptions required ensure that our results are broadly applicable to a wide range of quantum information tasks. Furthermore, we have provided explicit constructions of the optimal measurement for binary composite hypotheses and given an operational interpretation of the maximum overlap with free states in quantum resource theories. A full version of this paper is available at [24], which provides details and proofs of all results.

REFERENCES

- [1] E. Lehmann and J. P. Romano, *Testing statistical hypotheses*. 1 2022.
- [2] H. Chernoff, "A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations," *The Annals of Mathematical Statistics*, pp. 493–507, 1952.
- [3] K. M. Audenaert, J. Calsamiglia, R. Muñoz-Tapia, E. Bagan, L. Masanes, A. Acín, and F. Verstraete, "Discriminating states: The quantum Chernoff bound," *Physical Review Letters*, vol. 98, no. 16, p. 160501, 2007.
- [4] M. Nussbaum and A. Szkoła, "The Chernoff lower bound for symmetric quantum hypothesis testing," *The Annals of Statistics*, vol. 37, no. 2, pp. 1040–1057, 2009.
- [5] K. Li, "Discriminating quantum states: The multiple Chernoff distance," *The Annals of Statistics*, vol. 44, Aug. 2016.
- [6] F. G. Brandao and M. B. Plenio, "A generalization of quantum Stein's lemma," *Communications in Mathematical Physics*, vol. 295, no. 3, pp. 791–828, 2010.
- [7] F. G. Brandão, A. W. Harrow, J. R. Lee, and Y. Peres, "Adversarial hypothesis testing and a quantum Stein's lemma for restricted measurements," *IEEE Transactions on Information Theory*, vol. 66, no. 8, pp. 5037–5054, 2020.
- [8] M. Berta, F. G. Brandão, and C. Hirche, "On composite quantum hypothesis testing," *Communications in Mathematical Physics*, vol. 385, no. 1, pp. 55–77, 2021.
- [9] M. Mosonyi, Z. Szilagy, and M. Weiner, "On the error exponents of binary state discrimination with composite hypotheses," *IEEE Transactions on Information Theory*, vol. 68, p. 1032–1067, Feb. 2022.
- [10] M. Hayashi and H. Yamasaki, "The generalized quantum Stein's lemma and the second law of quantum resource theories," *Nature Physics*, pp. 1–6, 2025.
- [11] M. Hayashi and M. Tomamichel, "Correlation detection and an operational interpretation of the Rényi mutual information," *Journal of Mathematical Physics*, vol. 57, no. 10, 2016.
- [12] A. Dasgupta, N. A. Warsi, and M. Hayashi, "Universal tester for multiple independence testing and classical-quantum arbitrarily varying multiple access channel," *IEEE Transactions on Information Theory*, vol. 71, no. 5, pp. 3719–3765, 2025.
- [13] K. Fang, H. Fawzi, and O. Fawzi, "Adversarial quantum channel discrimination," *Physical Review Letters*, vol. 135, p. 260201, Dec 2025.
- [14] K. Watanabe and R. Takagi, "Black box work extraction and composite hypothesis testing," *Physical Review Letters*, vol. 133, no. 25, p. 250401, 2024.
- [15] F. Hiai, M. Mosonyi, and T. Ogawa, "Large deviations and chernoff bound for certain correlated states on a spin chain," *Journal of Mathematical Physics*, vol. 48, no. 12, 2007.
- [16] F. Hiai, M. Mosonyi, and T. Ogawa, "Error exponents in hypothesis testing for correlated states on a spin chain," *Journal of Mathematical Physics*, vol. 49, no. 3, 2008.
- [17] M. Mosonyi and T. Ogawa, "Two approaches to obtain the strong converse exponent of quantum hypothesis testing for general sequences of quantum states," *IEEE Transactions on Information Theory*, vol. 61, no. 12, pp. 6975–6994, 2015.
- [18] M. Hayashi and Y. Ito, "Entanglement measures for detectability," *IEEE Transactions on Information Theory*, vol. 71, p. 4385–4405, Apr. 2025.
- [19] K. Fang, H. Fawzi, and O. Fawzi, "Generalized quantum asymptotic equipartition," *arXiv preprint arXiv:2411.04035*, 2024.
- [20] L. Lami, "A solution of the generalized quantum Stein's lemma," *IEEE Transactions on Information Theory*, vol. 71, p. 4454–4484, June 2025.
- [21] M. Hayashi, "General detectability measure," *Communications in Mathematical Physics*, vol. 406, no. 12, p. 289, 2025.
- [22] L. Lami, M. Berta, and B. Regula, "Asymptotic quantification of entanglement with a single copy," *Nature Physics*, vol. 22, no. 3, pp. 439–445, 2026.
- [23] A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications*. Springer Berlin Heidelberg, 2010.
- [24] K. Fang and M. Hayashi, "Generalized quantum chernoff bound," *arXiv: 2508.12889*, 2025.
- [25] A. Holevo, "An analog of the theory of statistical decisions in non-commutative probability theory," *Trans. Moscow Math. Soc.*, vol. 26, no. 1972, pp. 133–149, 1972.
- [26] C. W. Helstrom, "Quantum detection and estimation theory," *Journal of Statistical Physics*, vol. 1, no. 2, pp. 231–252, 1969.
- [27] K. Fang and Z.-W. Liu, "No-go theorems for quantum resource purification," *Physical Review Letters*, vol. 125, no. 6, p. 060405, 2020.
- [28] K. Fang and Z.-W. Liu, "No-go theorems for quantum resource purification: New approach and channel theory," *PRX Quantum*, vol. 3, no. 1, p. 010337, 2022.
- [29] Z.-W. Liu, K. Bu, and R. Takagi, "One-shot operational quantum resource theory," *Physical Review Letters*, vol. 123, no. 2, p. 020401, 2019.
- [30] S. Bravyi, D. Browne, P. Calpin, E. Campbell, D. Gosset, and M. Howard, "Simulation of quantum circuits by low-rank stabilizer decompositions," *Quantum*, vol. 3, p. 181, 2019.