

Exponential Analysis for Entanglement Distillation

Zhiwen Lin

IASM, Harbin Institute of Technology
21b912029@stu.hit.edu.cn

Ke Li

IASM, Harbin Institute of Technology
carl.ke.lee@gmail.com

Kun Fang

kunfang00@outlook.com

Abstract—Historically, the focus in entanglement distillation has predominantly been on the distillable entanglement, and the framework assumes complete knowledge of the initial state. In this paper, we study the reliability function of entanglement distillation, which specifies the optimal exponent of the decay of the distillation error when the distillation rate is below the distillable entanglement. Furthermore, to capture greater operational significance, we extend the framework from the standard setting of known states to a black-box setting, where distillation is performed from a set of possible states. We establish an exact finite blocklength result connecting to composite correlated hypothesis testing without any redundant correction terms. Based on this, the reliability function of entanglement distillation is characterized by the regularized quantum Hoeffding divergence. In the special case of a pure initial state, our result reduces to the error exponent for entanglement concentration derived by Hayashi et al. in 2003. Given full prior knowledge of the state, we construct a concrete optimal distillation protocol. Additionally, we analyze the strong converse exponent of entanglement distillation. While all the above results assume the free operations to be non-entangling, we also investigate other free operation classes, including PPT-preserving, dually non-entangling, and dually PPT-preserving operations.

A full version of this paper is available at [38], which provides details and proofs of all results.

Index Terms—entanglement distillation, reliability function, strong converse exponent, non-entangling, distillation protocol.

I. INTRODUCTION

Entanglement was initially seen as a peculiar physical phenomenon [1]–[5], but is now regarded as a resource [6], [7]. It fuels quantum teleportation [8] and superdense coding [9], plays a critical role in quantum key distribution [10]–[12], and enhances quantum [13], [14] and private [15], [16] communications on noisy channels. Since most quantum communication schemes require maximally entangled states, it is essential to purify multiple copies of noisy, weakly entangled states into a smaller number of highly entangled states [17]–[19], a process termed *entanglement distillation*. Over the past two decades, significant attention has been focused on the asymptotic distillation rate, leading to breakthroughs under various free operations such as one-way local operations and classical communication [20], (asymptotically) non-entangling operations [21], [22], and dually non-entangling operations [23].

The *reliability function* of entanglement distillation is defined as the optimal error exponent. This optimality is considered within a specified class of free operations. It describes how fast the distillation error decays as the number of copies of the initial state increases, for a fixed distillation rate — that

is, the average number of target states that can be produced per initial copy. Hayashi et al. [24] explored the reliability function of entanglement distillation under local operations and classical communication (LOCC), but their study was limited to pure entangled states and is not applicable to mixed entangled states. Recently, Lami et al. [25] investigated the error exponent under non-entangling operations. However, their analysis fixed the total distillation yield rather than the distillation rate. Informally, their result is that the error exponent is given by the reverse relative entropy of entanglement when the distillation rate approaches zero.

In this paper, we study the reliability function of entanglement distillation to encompass not only mixed states but also the black-box scenario. In natural operational settings, complete knowledge of the initial state is unattainable due to unknown noise or adversarial effects. This information inaccessibility makes it essential to analyze the performance of entanglement distillation protocols independent of a specific state. Within the finite blocklength analysis, we derive an exact relationship between entanglement distillation and composite correlated hypothesis testing. To the best of our knowledge, this may be the first result that requires no additional correction terms. This fine-grained control of the error is crucial for analyzing the error exponent. Hence, we obtain that the reliability function of entanglement distillation is characterized by the regularized quantum Hoeffding divergence. When the initial state is pure, our framework yields the same error exponent formula for entanglement concentration as that established by Hayashi et al. [24]. Furthermore, our analysis holds for any fixed entanglement distillation rate below the distillable entanglement. For cases where the regularized Petz Rényi relative entropy with respect to the set of all separable states is continuous at $\alpha = 1$, our result provides an alternative approach to obtaining the result of Lami et al. [25] by taking the distillation rate to zero. Moreover, we construct an explicit optimal distillation protocol under the condition of complete prior information, which is guaranteed to achieve the reliability function.

In addition, We also establish that the *strong converse exponent* — defined as the optimal exponent for the decay of distillation fidelity when the rate exceeds the distillable entanglement — is lower-bounded by the regularized quantum Hoeffding anti-divergence. This bound becomes achievable when the initial state is known and the regularized sandwiched Rényi relative entropy is differentiable for $\alpha \geq 1$. Regarding free operations, we consider not only the well-studied

non-entangling operations, but also other operational classes including PPT-preserving, dually non-entangling, and dually PPT-preserving operations.

II. NOTATION AND PRELIMINARIES

A. Notation

A quantum system A is mathematically represented by a Hilbert space $\mathcal{H}_A$, with its dimension denoted by $|A|$. The set of positive semi-definite operators on $\mathcal{H}_A$ is denoted as $\mathcal{P}(\mathcal{H}_A)$. Quantum states of system A are modeled as density operators, i.e., positive semi-definite operators on $\mathcal{H}_A$ with trace one. Let $\mathcal{D}(\mathcal{H}_A)$ be the set of density operators on $\mathcal{H}_A$. Here, we shall consider the bipartite quantum system AB with the Hilbert space $\mathcal{H}_{AB} := \mathcal{H}_A \otimes \mathcal{H}_B$. A bipartite state σ_{AB} on AB is called *separable* [26] if it can be decomposed as the form $\sigma_{AB} = \sum_{i=1}^N p_i \alpha_A^i \otimes \beta_B^i$ where $N \in \mathbb{N}_+$, p is a probability distribution on $\{1, \dots, N\}$, α_A^i and β_B^i are states on systems A and B respectively. Otherwise it is called *entangled*, e.g., the two-qubit maximally entangled state $\Phi_2 := |\Phi_2\rangle\langle\Phi_2|$, where $|\Phi_2\rangle := \frac{1}{\sqrt{2}}|00\rangle + |11\rangle$. Let $\mathcal{F}_n$ denote the set of all separable states on the bipartite Hilbert space $\mathcal{H}_{AB}^{\otimes n}$, where separable partition is $A_1 \dots A_n : B_1 \dots B_n$. We denote by $\mathcal{F} = \{\mathcal{F}_n\}_{n \in \mathbb{N}}$ the sequence of separable state sets¹. A state ρ on Hilbert space $\mathcal{H}_{AB}$ is called a *maximally correlated state* if there exist bases $\{|a_i\rangle\}_i$ and $\{|b_i\rangle\}_i$ of $\mathcal{H}_A$ and $\mathcal{H}_B$ such that the support of ρ is contained in the subspace spanned by $\{|a_i\rangle|b_i\rangle\}_i$. Such a state is denoted by ρ^{mc} .

A quantum channel Λ is a completely positive and trace-preserving (CPTP) linear map acting on quantum states. A quantum measurement is mathematically represented by a set of positive semi-definite operators $\mathcal{M} = \{M_x\}_x$ satisfying the normalization condition $\sum_x M_x = \mathbb{1}$.

For $n \in \mathbb{N}$, let S_n be the symmetric group of the permutations of n elements. A state $\rho_n \in \mathcal{D}(\mathcal{H}^{\otimes n})$ is called permutation invariant if it satisfies $U_{\pi_n} \rho_n U_{\pi_n}^* = \rho_n, \forall \pi_n \in S_n$, where $U_{\pi_n} : |\psi_1\rangle \otimes \dots \otimes |\psi_n\rangle \mapsto |\psi_{\pi_n^{-1}(1)}\rangle \otimes \dots \otimes |\psi_{\pi_n^{-1}(n)}\rangle$ is the natural representation of permutation π_n .

For state ρ and σ on Hilbert space $\mathcal{H}$, the fidelity between them is defined as $F(\rho, \sigma) := \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$.

B. Quantum Divergences

A functional $\mathbb{D} : \mathcal{D} \times \mathcal{P} \rightarrow \mathbb{R}$ is called a *quantum divergence* if it satisfies the data-processing inequality: for any quantum channel Λ and any $(\rho, \sigma) \in \mathcal{D} \times \mathcal{P}$, it holds that $\mathbb{D}(\Lambda(\rho) \|\Lambda(\sigma)) \leq \mathbb{D}(\rho \|\sigma)$. In the following, we introduce several quantum divergences that will be used throughout this work. We also define quantum divergences between two sets of quantum states.

Definition 1: For any $(\rho, \sigma) \in \mathcal{D} \times \mathcal{P}$ and $\alpha \in (0, 1) \cup (1, \infty)$, the Umegaki relative entropy [27] is defined by

$$D(\rho \|\sigma) := \text{Tr}[\rho(\log \rho - \log \sigma)],$$

¹The notation $\mathcal{F}$ is abused to denote sets of states or sequences thereof, depending on context; likewise for $\mathcal{A}, \mathcal{B}$, and $\mathcal{R}$ in what follows.

the Petz Rényi divergence [28] is defined by

$$D_{\text{P}, \alpha}(\rho \|\sigma) := \frac{1}{\alpha - 1} \log \text{Tr} [\rho^\alpha \sigma^{1-\alpha}],$$

and the sandwiched Rényi divergence [29], [30] is defined by

$$D_{\text{S}, \alpha}(\rho \|\sigma) := \frac{1}{\alpha - 1} \log \text{Tr} \left[\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}} \right]^\alpha,$$

if either $\alpha < 1$ and $\rho \not\ll \sigma$ or $\alpha > 1$ and $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$. Otherwise, we set the divergences as $+\infty$.

Definition 2 (Quantum divergence between two sets of states): Let $\mathbb{D}$ be a quantum divergence between two quantum states. Then for any sets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{D}(\mathcal{H})$, the quantum divergence between these two sets is defined by

$$\mathbb{D}(\mathcal{A} \|\mathcal{B}) := \inf_{\rho \in \mathcal{A}, \sigma \in \mathcal{B}} \mathbb{D}(\rho \|\sigma).$$

Let $\mathcal{A} = \{\mathcal{A}_n\}_{n \in \mathbb{N}}$ and $\mathcal{B} = \{\mathcal{B}_n\}_{n \in \mathbb{N}}$ be two sequences of sets of quantum states, where each $\mathcal{A}_n, \mathcal{B}_n \subseteq \mathcal{D}(\mathcal{H}^{\otimes n})$. the regularized divergence between these sequences is defined by

$$\mathbb{D}^\infty(\mathcal{A} \|\mathcal{B}) := \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{D}(\mathcal{A}_n \|\mathcal{B}_n),$$

if the limit exists.

In many practical scenarios, the sequences of sets under consideration are not arbitrary but possess a structure that is compatible with tensor products. This property, known as *stability* (or *closeness*) under tensor product, is formalized as follows.

Definition 3 (Stable sequence): Let $\mathcal{A} \subseteq \mathcal{P}(\mathcal{H}_A), \mathcal{B} \subseteq \mathcal{P}(\mathcal{H}_B)$, and $\mathcal{C} \subseteq \mathcal{P}(\mathcal{H}_A \otimes \mathcal{H}_B)$. We say that $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is stable under tensor product if, for any $X_A \in \mathcal{A}$ and $X_B \in \mathcal{B}$, it holds that $X_A \otimes X_B \in \mathcal{C}$. In short, we write $\mathcal{A} \otimes \mathcal{B} \subseteq \mathcal{C}$. A sequence of sets $\{\mathcal{A}_n\}_{n \in \mathbb{N}}$ with $\mathcal{A}_n \subseteq \mathcal{P}(\mathcal{H}^{\otimes n})$ is called stable under tensor product if $\mathcal{A}_n \otimes \mathcal{A}_m \subseteq \mathcal{A}_{n+m}$ for all $n, m \in \mathbb{N}$.

C. Composite Correlated Hypothesis Testing

We set up a hypothesis testing problem with the composite null hypothesis comprising all separable states $\mathcal{F}_n$, against the alternative hypothesis which is the set of initial states $\mathcal{R}_n$. Two types of error can occur. The type-I error is the probability that we incorrectly accept the resource state $\mathcal{R}_n$ while it is actually the free state $\mathcal{F}_n$, and the type-II error corresponds to the converse situation. Providing a fixed number $r > 0$ and constraining the type-I error to be at most $2^{-\lfloor rn \rfloor}$, we denote the optimal type-II error by

$$\begin{aligned} & \beta_n(\mathcal{R}_n, 2^{-\lfloor rn \rfloor}) \\ & := \min_{\substack{0 \leq M_n \leq \mathbb{1} \\ \max_{\sigma_n \in \mathcal{F}_n} \text{Tr}((\mathbb{1} - M_n)\sigma_n) \leq 2^{-\lfloor rn \rfloor}}} \max_{\rho_n \in \mathcal{R}_n} \text{Tr} M_n \rho_n. \end{aligned}$$

III. PROBLEM STATEMENT AND MAIN RESULTS

Entanglement distillation involves two separated parties, conventionally named Alice and Bob. In previous studies [20]–[23], [25], two parties share multiple copies of a bipartite entangled state ρ_{AB} and aim to convert them into a smaller number of maximally entangled states. However, the initial

state is not fully accessible in practical implementations. Therefore, considering the initial resource states as a set $\mathcal{R}$, it becomes necessary to investigate universal distillation protocols in “state-agnostic” scenarios. For this conversion, not all quantum channels are available. We restrict the free operations to non-entangling operations, i.e., those that map separable states to separable states, denoted by $\mathcal{F}_{\text{NE}} := \{\Lambda \in \text{CTP} : \Lambda(\sigma) \in \mathcal{F}, \forall \sigma \in \mathcal{F}\}$.

A. The Reliability Function

Let $\mathcal{R}_n$ be a convex compact set of quantum states on Hilbert space $\mathcal{H}_{AB}^{\otimes n}$. When the distillation rate is r , which means that the yield is $\lfloor rn \rfloor$ copies of the two-qubit maximally entangled states $\Phi_2^{\otimes \lfloor rn \rfloor}$, the error of entanglement distillation in a black-box setting is defined as

$$\varepsilon(\mathcal{R}_n \xrightarrow{\mathcal{F}_{\text{NE}}} \Phi_2^{\otimes \lfloor rn \rfloor}) := \min_{\Lambda_n \in \mathcal{F}_{\text{NE}}} \max_{\rho_n \in \mathcal{R}_n} (1 - F(\Lambda_n(\rho_n), \Phi_2^{\otimes \lfloor rn \rfloor})).$$

In finite blocklength regime, by shifting our perspective from the only integer-valued and thus discrete yield to the continuously varying error, we establish an exact equivalence between the error of entanglement distillation in the black-box setting and the optimal type-II error in the composite correlated hypothesis testing.

Theorem 4 (Finite blocklength relation): Let $\mathcal{R}_n$ be a convex compact set of quantum states on Hilbert space $\mathcal{H}_{AB}^{\otimes n}$. Let $r > 0$ be a real number, and the distillation protocol is restricted to non-entangling operations $\mathcal{F}_{\text{NE}}$. Then the error of entanglement distillation in the block box setting equals the optimal type-II error of hypothesis testing

$$\varepsilon(\mathcal{R}_n \xrightarrow{\mathcal{F}_{\text{NE}}} \Phi_2^{\otimes \lfloor rn \rfloor}) = \beta_n(\mathcal{R}_n, 2^{-\lfloor rn \rfloor}). \quad (1)$$

We now turn to the asymptotic behavior of entanglement distillation. Let $\mathcal{R} = \{\mathcal{R}_n\}_{n \in \mathbb{N}}$ be a stable sequences of convex compact sets of quantum states, where $\mathcal{R}_n \subseteq \mathcal{D}(\mathcal{H}_{AB}^{\otimes n})$. When the distillation rate r is below the distillable entanglement, the error of entanglement distillation converges exponentially to 0. The reliability function of entanglement distillation is defined as the rate of such exponential decreasing,

$$E^{\text{rf}}(\mathcal{R}, \mathcal{F}_{\text{NE}}, r) := \lim_{n \rightarrow \infty} -\frac{1}{n} \log \varepsilon(\mathcal{R}_n \xrightarrow{\mathcal{F}_{\text{NE}}} \Phi_2^{\otimes \lfloor rn \rfloor}). \quad (2)$$

Based on the quantum Hoeffding bound for composite correlated hypotheses established in [31, Theorem 5.1], we immediately obtain the reliability function for entanglement distillation in the black-box setting.

Theorem 5 (Reliability function for entanglement distillation): Let $\mathcal{R} = \{\mathcal{R}_n\}_{n \in \mathbb{N}}$ be a stable sequences of convex compact sets of quantum states with each $\mathcal{R}_n \subseteq \mathcal{D}(\mathcal{H}_{AB}^{\otimes n})$. Let $0 < r < D^\infty(\mathcal{R} \parallel \mathcal{F})$ be a real number, and the distillation protocol is restricted to non-entangling operations $\mathcal{F}_{\text{NE}}$. Then the reliability function of entanglement distillation in the black-box setting is the regularized quantum Hoeffding divergences

$$E^{\text{rf}}(\mathcal{R}, \mathcal{F}_{\text{NE}}, r) = H_r^\infty(\mathcal{R} \parallel \mathcal{F}), \quad (3)$$

where the RHS follows Definition 2 and is induced by the Hoeffding divergence for two states

$$H_{n,r}(\rho_n \parallel \sigma_n) := \sup_{\alpha \in (0,1)} \frac{\alpha - 1}{\alpha} (nr - D_{\text{P},\alpha}(\rho_n \parallel \sigma_n)). \quad (4)$$

Remark 6: We observe that the distillation rate is associated with the Type-I error in hypothesis testing, while the distillation error is linked to the Type-II error. Furthermore, in the context of Hoeffding’s hypothesis testing framework, the Type-I error and the Type-II error are symmetric. Consequently, by restricting the Type-II error to 2^{-rn} and minimizing the Type-I error, the latter will converge exponentially at rate $H_r^\infty(\mathcal{F} \parallel \mathcal{R})$. Let the parameter r decay as $O(1/n)$, then it immediately follows that the distillable entanglement in the black-box setting is bounded by the regularized relative entropy of composite entanglement $D^\infty(\mathcal{R} \parallel \mathcal{F})$. Provided that the regularized Petz Rényi relative entropy $D_{\text{P},\alpha}^\infty(\mathcal{R} \parallel \mathcal{F})$ exhibits continuity at $\alpha = 1$, the distillation rate is characterized by $D^\infty(\mathcal{R} \parallel \mathcal{F})$.

Given complete prior knowledge of the initial state—that is, when distillation starts from a precisely known quantum state ρ_{AB} —setting $\mathcal{R}_n = \rho^{\otimes n}$ immediately yields a corollary.

Corollary 7: Let ρ be an entangled state on Hilbert space $\mathcal{H}_{AB}$. Let $0 < r < D^\infty(\rho \parallel \mathcal{F})$ be a real number, and the distillation protocol is restricted to non-entangling operations $\mathcal{F}_{\text{NE}}$. Then the reliability function of entanglement distillation with complete information about the initial state is the regularized quantum Hoeffding divergences

$$E^{\text{rf}}(\rho, \mathcal{F}_{\text{NE}}, r) = H_r^\infty(\rho \parallel \mathcal{F}). \quad (5)$$

Remark 8: When the initial state is a maximally correlated state ρ^{mc} , due to the additivity of the Petz Rényi relative entropy with respect to the set of separable states [32], the reliability function of entanglement distillation has a single-letter expression,

$$E^{\text{rf}}(\rho^{\text{mc}}, \mathcal{F}_{\text{NE}}, r) = \sup_{\alpha \in (0,1)} \frac{\alpha - 1}{\alpha} (r - D_{\text{P},\alpha}(\rho^{\text{mc}} \parallel \mathcal{F}_1)). \quad (6)$$

Note that maximally correlated states include all pure states. Moreover, the errors of entanglement distillation for non-entangling and LOCC operations coincide in the distillation of pure entangled states [33]. By leveraging an alternative expression of the Petz Rényi relative entropy of entanglement [32, Corollary 2], we can reproduce the result of Hayashi et al. [24].

Remark 9: The framework of [25] employs non-entangling operations to distill entanglement from a state ρ_{AB} , establishing that the error exponent in the vanishing-rate (i.e., $r \rightarrow 0$) is the reverse relative entropy of entanglement $D(\mathcal{F}_1 \parallel \rho)$. Taking the limit as $r \rightarrow 0$ in our result immediately yields that the error exponent is upper bounded by $D(\mathcal{F}_1 \parallel \rho)$. Moreover, provided that the continuity of the regularized Petz Rényi divergence holds at $\alpha = 1$, i.e., $\sup_{\alpha \in (0,1)} D_{\text{P},\alpha}^\infty(\mathcal{F} \parallel \rho) = D^\infty(\mathcal{F} \parallel \rho)$, our result would recover the finding of [25]. See more discussions in Section IV.

B. The Strong Converse Exponent

Returning to the black-box scenario, let $\mathcal{R} = \{\mathcal{R}_n\}_{n \in \mathbb{N}}$ be a stable sequences of convex compact sets of quantum states with each $\mathcal{R}_n \subseteq \mathcal{D}(\mathcal{H}_{AB}^{\otimes n})$. When the distillation rate r exceeds the distillable entanglement, the error of entanglement distillation no longer tends to 0 but instead inevitably converges to 1, as shown in Proposition 11 below. To precisely describe the asymptotic behavior in this regime, we define the fidelity of entanglement distillation in the black-box setting,

$$f(\mathcal{R}_n \xrightarrow{\mathcal{F}_{\text{NE}}} \Phi_2^{\otimes \lfloor rn \rfloor}) := \max_{\Lambda_n \in \mathcal{F}_{\text{NE}}} \min_{\rho_n \in \mathcal{R}_n} F(\Lambda_n(\rho_n), \Phi_2^{\otimes \lfloor rn \rfloor}).$$

The strong converse exponent for entanglement distillation in black-box setting is defined as the convergence exponent of the fidelity of entanglement distillation,

$$E^{\text{sc}}(\mathcal{R}, \mathcal{F}_{\text{NE}}, r) := \liminf_{n \rightarrow \infty} -\frac{1}{n} \log f(\mathcal{R}_n \xrightarrow{\mathcal{F}_{\text{NE}}} \Phi_2^{\otimes \lfloor rn \rfloor}). \quad (7)$$

A direct observation from the definition is that the fidelity and the error of entanglement distillation in the black-box setting satisfy the relation

$$f(\mathcal{R}_n \xrightarrow{\mathcal{F}_{\text{NE}}} \Phi_2^{\otimes \lfloor rn \rfloor}) = 1 - \varepsilon(\mathcal{R}_n \xrightarrow{\mathcal{F}_{\text{NE}}} \Phi_2^{\otimes \lfloor rn \rfloor}).$$

Similar to the approach used in proving the reliability function, it is useful to establish a connection with hypothesis testing. Therefore, we consider the *detection probability* [34] of $\mathcal{R}_n$ under the null hypothesis $\mathcal{F}_n$ at a level $2^{-\lfloor rn \rfloor}$,

$$\begin{aligned} & 1 - \beta_n(\mathcal{R}_n, 2^{-\lfloor rn \rfloor}) \\ &= \max_{\substack{0 \leq M_n \leq \mathbb{I} \\ \max_{\sigma_n \in \mathcal{F}_n} \text{Tr}((\mathbb{I} - M_n)\sigma_n) \leq 2^{-\lfloor rn \rfloor}}} \min_{\rho_n \in \mathcal{R}_n} (1 - \text{Tr}(M_n \rho_n)). \end{aligned}$$

From Theorem 4, we immediately derive the following corollary.

Corollary 10: Let $\mathcal{R}_n$ be a convex compact set of quantum states on Hilbert space $\mathcal{H}_{AB}^{\otimes n}$. Let $r > 0$ be a real number, and the distillation protocol is restricted to non-entangling operations $\mathcal{F}_{\text{NE}}$. Then the fidelity of entanglement distillation in the black-box setting can be related to the detection probability of hypothesis testing

$$f(\mathcal{R}_n \xrightarrow{\mathcal{F}_{\text{NE}}} \Phi_2^{\otimes \lfloor rn \rfloor}) = 1 - \beta_n(\mathcal{R}_n, 2^{-\lfloor rn \rfloor}). \quad (8)$$

Applying the strong converse exponent for composite correlated hypotheses established in [31, Theorem 6.1] yields the following results for entanglement distillation.

Proposition 11: Let $\mathcal{R} = \{\mathcal{R}_n\}_{n \in \mathbb{N}}$ be a stable sequences of convex compact sets of quantum states with each $\mathcal{R}_n \subseteq \mathcal{D}(\mathcal{H}_{AB}^{\otimes n})$. Let $r > D^\infty(\mathcal{R} \parallel \mathcal{F})$ be a real number, and the distillation protocol is restricted to non-entangling operations $\mathcal{F}_{\text{NE}}$. Then the strong converse exponent for entanglement distillation in the black-box setting is lower bounded by the regularized quantum Hoeffding anti-divergences

$$E^{\text{sc}}(\mathcal{R}, \mathcal{F}_{\text{NE}}, r) \geq \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} (r - D_{s, \alpha}^\infty(\mathcal{R} \parallel \mathcal{F})). \quad (9)$$

Furthermore, if $\{\mathcal{R}_n\}_{n \in \mathbb{N}} = \{\rho_{AB}^{\otimes n}\}_{n \in \mathbb{N}}$ and $D_{s, \alpha}^\infty(\mathcal{R} \parallel \mathcal{F})$ is differentiable for $\alpha \geq 1$, then the strong converse exponent

for entanglement distillation with complete information about the initial state is characterized by the regularized quantum Hoeffding anti-divergences

$$E^{\text{sc}}(\rho, \mathcal{F}_{\text{NE}}, r) = \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} (r - D_{s, \alpha}^\infty(\rho \parallel \mathcal{F})). \quad (10)$$

C. Error Exponents across Different Free Operations

Within the finite blocklength framework, the proof approach of Theorem 4 can be extended directly to PPT states and PPT-preserving operations upon replacing separable states and non-entangling operations. In addition, PPT states satisfy the conditions required for composite correlated hypothesis testing. Consequently, the reliability function for entanglement distillation under PPT-preserving operations can be derived.

Example 1 (PPT states and PPT-preserving operations): A state σ on Hilbert space $\mathcal{H}_{AB}$ is called the positive-partial-transpose (PPT) state if it satisfies $\sigma_{AB}^\Gamma \geq 0$, where Γ denotes the partial transpose. The set of PPT states is denoted by PPT. The class of positive-partial-transpose-preserving (PPT-preserving) operations is denoted as $\mathcal{F}_{\text{PPT}} := \{\Lambda \in \text{CPTP} : \Lambda(X) \in \text{PPT}, \forall X \in \text{PPT}\}$.

Then we get the error of entanglement distillation under PPT-preserving operations corresponds precisely to the Type-II error of hypothesis testing with null hypothesis PPT and alternative hypothesis $\mathcal{R}$,

$$\varepsilon(\mathcal{R}_n \xrightarrow{\mathcal{F}_{\text{PPT}}} \Phi_2^{\otimes \lfloor rn \rfloor}) = \beta_n(\mathcal{R}_n, \text{PPT}, 2^{-\lfloor rn \rfloor}). \quad (11)$$

The reliability function of entanglement distillation under PPT-preserving operations is characterized by a regularized quantum Hoeffding divergence

$$E^{\text{rf}}(\mathcal{R}, \mathcal{F}_{\text{PPT}}, r) = H_r^\infty(\mathcal{R} \parallel \text{PPT}). \quad (12)$$

Remark 12: Consider the distillation rate $r = m/n$ for some fixed integer m and the resource sequence $\{\mathcal{R}_n\}_{n \in \mathbb{N}} = \{\rho_{AB}^{\otimes n}\}_{n \in \mathbb{N}}$. Applying the approach of [25] or Theorem 10 of [34] to the known initial state ρ_{AB} in the vanishing rate limit, we obtain that the error exponent is characterized as the single-letter divergence $D(\text{PPT} \parallel \rho)$ under PPT-preserving operations. This achieves a result analogous to that in [25].

Except for the constraint on state evolution in the Schrödinger picture, specifically non-entangling and PPT-preserving operations, we additionally impose constraints on the evolution of measurement operators in the Heisenberg picture, thereby defining the dually non-entangling and dually PPT-preserving operations.

Example 2 (Separable states and dually non-entangling operations): A channel is called dually non-entangling if it is non-entangling in the Schrödinger picture, i.e., it maps separable states to separable states, and its adjoint is non-entangling in the Heisenberg picture, i.e., it maps separable measurement operators to separable operators. The class of dually non-entangling operations is denoted as $\mathcal{F}_{\text{DNE}} := \{\Lambda \in \text{CPTP} : \Lambda(\sigma) \in \mathcal{F}, \forall \sigma \in \mathcal{F}, \text{ and } \Lambda^\dagger(M) \in \text{cone}(\mathcal{F}), \forall M \in \text{cone}(\mathcal{F})\}$, where $\text{cone}(\mathcal{F})$ is the cone generated by the set of separable states $\mathcal{F}$.

To address this scenario, we formulate a hypothesis testing framework restricted to separable measurements SEP for discriminating between resource states $\mathcal{R}$ and separable states. When the sequence of initial state sets $\{\mathcal{R}_n\}_{n \in \mathbb{N}}$ is compatible with separable measurements and the distillation rate approaches zero, by applying Theorem 16 of [35], we obtain the error exponent is given by

$$D^{\text{SEP}, \infty}(\mathcal{F} \parallel \rho) := \lim_{n \rightarrow \infty} \frac{1}{n} D^{\text{SEP}}(\mathcal{F}_n \parallel \rho^{\otimes n}), \quad (13)$$

where

$$D^{\text{SEP}}(\sigma \parallel \rho) := \sup_{\{M_x\}_x \in \text{SEP}} \sum_x \text{Tr}(M_x \sigma) \log \frac{\text{Tr}(M_x \sigma)}{\text{Tr}(M_x \rho)}.$$

Example 3 (PPT states and dually PPT-preserving operations): A channel is dually PPT-preserving if it preserves PPT states in the Schrödinger picture and its adjoint preserves PPT measurement operators in the Heisenberg picture. We denote the class of dually PPT-preserving operations by $\mathcal{F}_{\text{DPPT}} := \{\Lambda \in \text{CPTP} : \Lambda(\sigma) \in \text{PPT}, \forall \sigma \in \text{PPT}, \text{ and } \Lambda^\dagger(M) \in \text{cone}(\text{PPT}), \forall M \in \text{cone}(\text{PPT})\}$, where $\text{cone}(\text{PPT})$ is the cone generated by the set of PPT states.

Analogous to Example 2, this is resolved by formulating a hypothesis testing framework with PPT measurements PPT to discriminate between resource states $\mathcal{R}_n$ and PPT states. When the sequence of initial state sets $\{\mathcal{R}_n\}_{n \in \mathbb{N}}$ is compatible with PPT measurements, the distillation operations are restricted to be dually PPT-preserving and the distillation rate approaches zero, then the error exponent for entanglement distillation is characterized by

$$D^{\text{PPT}, \infty}(\text{PPT} \parallel \rho) := \lim_{n \rightarrow \infty} \frac{1}{n} D^{\text{PPT}}(\text{PPT}_n \parallel \rho^{\otimes n}), \quad (14)$$

where

$$D^{\text{PPT}}(\sigma \parallel \rho) := \sup_{\{M_x\}_x \in \text{PPT}} \sum_x \text{Tr}(M_x \sigma) \log \frac{\text{Tr}(M_x \sigma)}{\text{Tr}(M_x \rho)}.$$

D. Optimal Distillation Protocol

In this section, we develop a concrete distillation protocol, based on the information-spectrum method, for the case of complete prior knowledge of the initial entangled state ρ_{AB} . This protocol achieves the optimal convergence exponent given in (5).

The construction of the explicit distillation protocol relies on the existence of a universal permutation invariant separable state. Note that this does not contradict the non-existence conclusion of [36], as the present work additionally allows entanglement among $A_1 \dots A_n$ and likewise among $B_1 \dots B_n$. This lemma is adapted from Lemma 1 of [37].

Lemma 13: In the Hilbert space $\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1} \otimes \dots \otimes \mathcal{H}_{A_n} \otimes \mathcal{H}_{B_n}$, where $\mathcal{H}_{A_i} = \mathcal{H}_A$ and $\mathcal{H}_{B_i} = \mathcal{H}_B$ for each i , there exists a universal permutation invariant product state $\omega_{A^n:B^n}^{(n)}$, such that for any permutation invariant separable state $\sigma_{A^n:B^n}$, we have $\sigma_{A^n:B^n} \leq g_{n,|A|,|B|} \omega_{A^n:B^n}^{(n)}$ where $g_{n,|A|,|B|} \leq (n+1)^{|A|^2+|B|^2-2}$. The above separable partition is $A_1 \dots A_n : B_1 \dots B_n$, and the permutation acts on the n subspaces $A_1 B_1 : \dots : A_n B_n$.

We now construct an entanglement distillation protocol that attains the optimal error exponent.

Theorem 14: Let ρ be an entangled state on Hilbert space $\mathcal{H}_{AB}$. The entanglement distillation protocol is constructed by

$$\Lambda_n(X) = \text{Tr}((\mathbb{1} - M_n)X) \Phi_2^{\otimes \lfloor rn \rfloor} + \text{Tr}(M_n X) \frac{\mathbb{1} - \Phi_2^{\otimes \lfloor rn \rfloor}}{(2^{\lfloor rn \rfloor})^2 - 1},$$

where $M_n = \{\rho^{\otimes n} \leq 2^{a_n} \omega_{A^n:B^n}^{(n)}\}$, i.e., the projection onto the positive part of the operator $2^{a_n} \omega_{A^n:B^n}^{(n)} - \rho^{\otimes n}$ and

$$a_n = \frac{rn + (s-1)D_{\text{P},s}(\rho^{\otimes n} \parallel \mathcal{F}_n) + \log g_{n,|A|,|B|} + 1}{s},$$

for a fixed $s \in (0, 1)$. Then the protocol is non-entangling and can achieve the reliability function of entanglement distillation with the initial state ρ in Corollary 7.

IV. CONCLUSION AND DISCUSSION

Our black-box extension of the conventional entanglement distillation is a crucial step towards modelling realistic scenario, accounting for both environmental uncertainty and non-i.i.d. correlations. In particular, we establish an exact equivalence, free of redundant correction terms, between the error of entanglement distillation in the black-box setting and the optimal type-II error in composite correlated hypothesis testing. Therefore, we obtain the precise reliability function for entanglement distillation in the black-box setting under the classes of non-entangling and PPT-preserving operations. When the initial state is pure, our result reduces to the formula of Hayashi et al. [24]. We also derive the error exponent for dually non-entangling and dually PPT-preserving operations in the limit of vanishing distillation rate. Given complete knowledge of the initial state, we construct an explicit non-entangling protocol that achieves the reliability function. Additionally, a lower bound for the strong converse exponent of entanglement distillation is provided in the black-box setting.

Investigating the continuity of the regularized Petz Rényi relative entropy at $\alpha = 1$ is an interesting question, as it relates to recovering the result of Lami et al. [25] and to the distillable entanglement in the black-box setting. Although [31] provides counterexamples demonstrating the breakdown of continuity in the general case, no explicit counterexample has been identified for scenarios where the first component constitutes sequences of separable state sets. Notably, the antisymmetric Werner state—the universal counterexample in entanglement theory—remains continuous under this configuration [31]. Moreover, extending the study of the reliability function to the distillation of multipartite entanglement and other resources presents an interesting direction for future work.

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