On finite blocklength converse bounds for classical communication over quantum channels

Xin Wang*, Kun Fang*, Marco Tomamichel*
*Centre for Quantum Software and Information
Faculty of Engineering and Information Technology
University of Technology Sydney, NSW 2007, Australia
Email: {Xin.Wang-8, Kun.Fang-1}@student.uts.edu.au, Marco.Tomamichel@uts.edu.au

***Abstract*—We explore several new converse bounds for classical communication over quantum channels in the finite blocklength regime. First, we show that the Matthews-Wehner meta-converse bound for entanglement-assisted classical communication can be achieved by activated, no-signalling assisted codes, suitably generalizing a result for classical channels. Second, we derive a new meta-converse on the amount of information unassisted codes can transmit over a single use of a quantum channel. We further show that this meta-converse can be evaluated via semidefinite programming. As an application, we provide a second-order analysis of classical communication over quantum erasure channels.**

I. Introduction

The central problem in quantum information theory is to determine the capability of a noisy quantum channel to transmit classical messages faithfully. The classical capacity of a quantum channel is the highest rate (in bits per channel use) at which it can convey classical information such that the error probability vanishes asymptotically as the code length increases. The Holevo-Schumacher-Westmoreland (HSW) theorem [1]–[3] establishes that the classical capacity of a noisy quantum channel is given by its regularized Holevo information. For certain classes of quantum channels (e.g., depolarizing channel [4], erasure channel [5], unital qubit channel [6], etc. [7]–[10]), the Holevo capacity is known to be additive and regularization is thus unnecessary; however, this is not true in general [11].

However, in realistic settings there are natural restrictions imposed on the code length. One fundamental question thus asks how much classical information can be transmitted over a single use of a quantum channel when a finite decoding error is tolerated. Several upper and lower bounds on this one-shot quantity were explored, e.g. in [12]–[17], but these in general do not match and are often hard to compute.

In Section III we build on an exact expression for the amount of classical information that can be transmitted over a single use of a quantum channel using codes that are assisted by no-signalling correlations provided in [17]. Using this result we show that the hypothesis testing relative entropy converse bound by Matthews and Wehner [16] can be achieved and is optimal for activated, no-signalling assisted codes. This generalizes a result by Matthews [18] for no-signalling assisted classical codes to the quantum setting, with the additional twist that in the quantum setting the codes require a classical noiseless channel as a catalyst.

Our Section IV provides a new meta-converse that upper bounds the amount of information that can be transmitted with a single use of the channel by unassisted codes. This meta-converse, in the spirit of the classical meta-converse by Polyanskiy, Poor and Verdú [19] as well as Nagaoka and Hayashi (see, e.g., [20], [15, Section 4.6]), relates the channel coding problem to a binary composite hypothesis test between the actual channel and a class of subchannels that are generalizations of the useless channels for classical communication. As a simple application we apply our meta-converse to establish second-order asymptotics [21] of the quantum erasure channel.

II. Unassisted, entanglement-assisted and no-signalling assisted codes

For our purposes, a quantum channel $\mathcal{N}_{A'\to B}$ is a completely positive (CP) and trace-preserving (TP) linear map from operators on a finite-dimensional Hilbert space A' to operators on a finite-dimensional Hilbert space B. Alice wants to send the classical messages to Bob using the composite channel $\mathcal{M}_{A\to B'} = \Pi_{AB\to A'B'} \circ \mathcal{N}_{A'\to B}$, where Π is a quantum bipartite operation that generalizes the usual encoding scheme $\mathcal{E}$ and decoding scheme $\mathcal{D}$. We say such Π is an Ω-assisted code if it can be implemented by local operations with Ω-assistance. In the following, we eliminate Ω for the case of unassisted codes and write $\Omega = \mathrm{E}$ and $\Omega = \mathrm{NS}$ for entanglement-assisted and no-signalling-assisted (NS-assisted) codes, respectively. The NS-assisted codes have also been applied to study the various kinds of communication tasks with quantum systems (e.g., [22]–[29]).

In particular,

- an unassisted code reduces to the product of encoder and decoder, i.e., $\Pi = \mathcal{D}_{B\to B'}\mathcal{E}_{A\to A'}$;
- an entanglement-assisted code corresponds to a bipartite operation of the form $\Pi = \mathcal{D}_{B\widehat{B}\to B'}\mathcal{E}_{A\widehat{A}\to A'}\Psi_{\widehat{A}\widehat{B}}$, where $\Psi_{\widehat{A}\widehat{B}}$ can be any entangled state shared between Alice and Bob;
- a NS-assisted code corresponds to a bipartite operation which is no-signalling from Alice to Bob and vice-versa.

978-1-5386-4780-6/18/$31.00©2018 IEEE

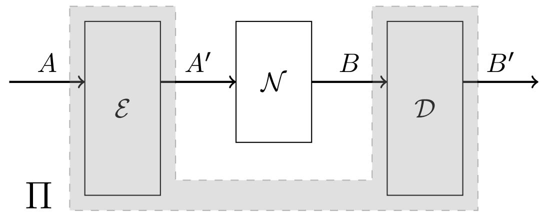

Fig. 1. General code scheme

Given a quantum channel $\mathcal{N}_{A\to B}$ and any Ω-assisted code Π with size m, the optimal average success probability of $\mathcal{N}$ to transmit m messages is given by

$$p_{\text{succ},\Omega}(\mathcal{N},m) := \frac{1}{m}\sup\sum_{k=1}^{m}\operatorname{Tr}\mathcal{M}(|k\rangle\langle k|)|k\rangle\langle k|,$$
$$\text{s.t. } \mathcal{M}=\Pi\circ\mathcal{N} \text{ is the effective channel.} \tag{1}$$

With this in hand, we now say that a triplet (r,n,ε) is achievable on the channel $\mathcal{N}$ with Ω-assisted codes if

$$\frac{1}{n}\log m \geq r, \text{ and } p_{\text{succ},\Omega}(\mathcal{N}^{\otimes n},m)\geq 1-\varepsilon. \tag{2}$$

We are interested in the following boundary of the non-asymptotic achievable region:

$$C_{\Omega}^{(1)}(\mathcal{N},\epsilon) := \sup\{\log m : p_{\text{succ},\Omega}(\mathcal{N},m)\geq 1-\varepsilon\}. \tag{3}$$

We also define $p_{\text{succ},\Omega}(\mathcal{N},\rho_A,m)$ and $C_{\Omega}^{(1)}(\mathcal{N},\rho_A,\epsilon)$ as the same optimization but only using codes with a fixed average input ρ_A. The Ω-assisted capacity of a quantum channel is

$$C_{\Omega}(\mathcal{N}) = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}C_{\Omega}^{(1)}(\mathcal{N}^{\otimes n},\epsilon). \tag{4}$$

III. Matthews-Wehner converse via activated, no-signalling assisted codes

For classical communication over quantum channels assisted by entanglement, Matthews and Wehner [16] proved a meta-converse bound in terms of the hypothesis testing relative entropy which generalizes Polyanskiy, Poor and Verdú's approach [19] to quantum channels assisted by entanglement. Given a quantum channel $\mathcal{N}$, they proved [16] that

$$C_E^{(1)}(\mathcal{N},\varepsilon) \leq R(\mathcal{N},\epsilon) := \max_{\rho_{A'}}\min_{\sigma_B} D_H^{\epsilon}(\mathcal{N}_{A\to B}(\phi_{A'A})\|\rho_{A'}\otimes\sigma_B), \tag{5}$$

where $\phi_{AA'} = (\mathbb{1}_A\otimes\rho_{A'}^{1/2})\widetilde{\Phi}_{AA'}(\mathbb{1}_A\otimes\rho_{A'}^{1/2})$ is a purification of $\rho_{A'}$ and $\widetilde{\Phi}_{AA'} = \sum_{ij}|i_Ai_{A'}\rangle\langle j_Aj_{A'}|$ denotes the unnormalized maximally entangled state. In the above expression the quantum hypothesis testing relative entropy [13] is defined as

$$D_H^{\epsilon}(\rho_0\|\rho_1) := -\log\beta_{\epsilon}(\rho_0\|\rho_1) \tag{6}$$
$$= -\log\min\{\operatorname{Tr}Q\rho_1 : 1-\operatorname{Tr}Q\rho_0\leq\epsilon, 0\leq Q\leq\mathbb{1}\},$$

where $\beta_{\epsilon}(\rho_0\|\rho_1)$ is the minimum type-II error for the test while the type-I error is no greater than ϵ. Note that β_{ϵ} is a fundamental quantity in quantum theory [30]–[32] and can be solved by a semidefinite program (SDP). The hypothesis testing relative entropy bound in Eq. (5) thus constitutes an SDP itself, i.e.

$$\begin{aligned} R(\mathcal{N},\epsilon) = -\log\min\ &\lambda \\ \text{s.t. } &0\leq F_{AB}\leq\rho_A\otimes\mathbb{1}_B, \\ &\operatorname{Tr}\rho_A = 1, \\ &\operatorname{Tr}_A F_{AB}\leq\lambda\mathbb{1}_B, \\ &\operatorname{Tr}J_{\mathcal{N}}F_{AB}\geq 1-\epsilon. \end{aligned} \tag{7}$$

Here the Choi-Jamiołkowski matrix [33], [34] of $\mathcal{N}$ is given by $J_{\mathcal{N}} = \sum_{ij}|i_A\rangle\langle j_A|\otimes\mathcal{N}(|i_{A'}\rangle\langle j_{A'}|)$, where $\{|i_A\rangle\}$ and $\{|i_{A'}\rangle\}$ are orthonormal bases on isomorphic Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_{A'}$, respectively. Also, note that SDP can be solved efficiently in general [35] and has many applications in quantum information theory.

For classical channels, the hypothesis testing relative entropy bound is exactly equal to the one-shot classical capacity assisted by no-signalling (NS) codes [18]. For quantum channels the one-shot ϵ-error capacity assisted by NS codes is given by [17]

$$\begin{aligned} C_{\text{NS}}^{(1)}(\mathcal{N},\epsilon) = -\log\min\ &\eta \\ \text{s.t. } &0\leq F_{AB}\leq\rho_A\otimes\mathbb{1}_B, \\ &\operatorname{Tr}\rho_A = 1, \\ &\operatorname{Tr}_A F_{AB} = \eta\mathbb{1}_B, \\ &\operatorname{Tr}J_{\mathcal{N}}F_{AB}\geq 1-\epsilon. \end{aligned} \tag{8}$$

Note that the only difference between the SDPs (7) and (8) is the partial trace constraint of F_{AB}. However, unlike in the classical special case, the SDPs in (7) and (8) are not equal in general [17].

In this section we show that this gap can be closed by considering activated, NS-assisted codes. The concept of activated capacity follows the idea of potential capacities of quantum channels introduced by Winter and Yang [36]. The model is described as follows. For a quantum channel $\mathcal{N}$ assisted by NS codes, we can first borrow a noiseless classical channel $\mathcal{I}_m$ whose capacity is $\log m$, then we can use $\mathcal{N}\otimes\mathcal{I}_m$ coherently to transmit classical messages. After the communication finishes, we just pay back the capacity of $\mathcal{I}_m$. This kind of scenario was also studied in zero-error information theory [37], [38].

Definition 1 *For any quantum channel $\mathcal{N}$, we define*

$$C_{\text{NS,a}}^{(1)}(\mathcal{N},\epsilon) := \sup_{m\geq 1}\Big[C_{\text{NS}}^{(1)}(\mathcal{N}\otimes\mathcal{I}_m,\epsilon) - \log m\Big], \tag{9}$$

where $\mathcal{I}_m(\rho) = \sum_{i=1}^{m}\operatorname{Tr}(\rho|i\rangle\langle i|)|i\rangle\langle i|$ the classical noiseless channel with capacity $\log m$.

The following is the main result of this section:

Theorem 2 *For any quantum channel $\mathcal{N}_{A\to B}$ and error tolerance $\varepsilon\in(0,1)$, we have*

$$C_{\text{NS,a}}^{(1)}(\mathcal{N},\epsilon) = R(\mathcal{N},\epsilon) \tag{10}$$
$$= \max_{\rho_{A'}}\min_{\sigma_B} D_H^{\epsilon}(\mathcal{N}_{A\to B}(\phi_{A'A})\|\rho_{A'}\otimes\sigma_B). \tag{11}$$

The proof outline is as follows. We first show that the $\mathcal{I}_2$ is enough to activate the channel to achieve the bound $R(\mathcal{N},\epsilon)$ in the following Lemma 3, i.e.,

$$C^{(1)}_{\mathrm{NS,a}}(\mathcal{N},\epsilon) \geq C^{(1)}_{\mathrm{NS}}(\mathcal{N}\otimes\mathcal{I}_2,\epsilon)-1 \quad (12)$$
$$\geq R(\mathcal{N},\epsilon). \quad (13)$$

We then show that $R(\mathcal{N},\epsilon)$ is additive for noiseless channel in the following Lemma 4, i.e., $R(\mathcal{N}\otimes\mathcal{I}_m,\epsilon)=R(\mathcal{N},\epsilon)+\log m$. This implies that $R(\mathcal{N},\epsilon)$ is also a converse bound for the activated capacity, i.e.,

$$C^{(1)}_{\mathrm{NS,a}}(\mathcal{N},\epsilon) = \sup_{m\geq 1}\left[C^{(1)}_{\mathrm{NS}}(\mathcal{N}\otimes\mathcal{I}_m,\epsilon)-\log m\right] \quad (14)$$
$$\leq \sup_{m\geq 1}\left[R(\mathcal{N}\otimes\mathcal{I}_m,\epsilon)-\log m\right]=R(\mathcal{N},\epsilon). \quad (15)$$

The theorem thus directly follows from Lemmas 3 and 4.

Lemma 3 *We have* $C^{(1)}_{\mathrm{NS}}(\mathcal{N}\otimes\mathcal{I}_2,\epsilon)-1\geq R(\mathcal{N},\epsilon)$.

Proof: This proof is based on a key observation that the additional one-bit noiseless channel can provide a larger solution space to help the activated capacity achieve the quantum hypothesis testing converse. Suppose that the optimal solution to SDP (7) of $R(\mathcal{N},\epsilon)$ is $\{\lambda,\rho_{A_1},F_{A_1B_1}\}$. We are going to use this optimal solution to construct a feasible solution of the SDP (8) of $C^{(1)}_{\mathrm{NS}}(\mathcal{N}\otimes I_2,\epsilon)$.

Let us choose $\rho_{A_1A_2}=\rho_{A_1}\otimes\frac{1}{2}(|0\rangle\langle 0|+|1\rangle\langle 1|)_{A_2}$ and

$$F_{A_1A_2B_1B_2} = \frac{F_{A_1B_1}}{2}\otimes(|00\rangle\langle 00|+|11\rangle\langle 11|)_{A_2B_2} \quad (16)$$
$$+\frac{\widetilde{F}_{A_1B_1}}{2}\otimes(|01\rangle\langle 01|+|10\rangle\langle 10|)_{A_2B_2}, \quad (17)$$

where $\widetilde{F}_{A_1B_1}=\rho_{A_1}\otimes(\lambda\mathbb{1}_{B_1}-\operatorname{Tr}_{A_1}F_{A_1B_1})$. We see that $F_{A_1A_2B_1B_2}\geq 0$, $\rho_{A_1A_2}\geq 0$ and $\operatorname{Tr}\rho_{A_1A_2}=1$. Moreover, this construction ensures that

$$\operatorname{Tr}_{A_1A_2}F_{A_1A_2B_1B_2} = \operatorname{Tr}_{A_1}\left(\left(\frac{F_{A_1B_2}}{2}+\frac{\widetilde{F}_{A_1B_1}}{2}\right)\otimes\mathbb{1}_{B_2}\right) \quad (18)$$
$$=\frac{\lambda}{2}\mathbb{1}_{B_1B_2}, \quad (19)$$

and

$$\operatorname{Tr}(J_{\mathcal{N}}\otimes D_{A_2B_2})F_{A_1A_2B_1B_2} \quad (20)$$
$$=\operatorname{Tr}J_{\mathcal{N}}F_{A_1B_1}\otimes\frac{1}{2}\operatorname{Tr}D_{A_2B_2}(|00\rangle\langle 00|+|11\rangle\langle 11|) \quad (21)$$
$$=\operatorname{Tr}J_{\mathcal{N}}F_{A_1B_1}\geq 1-\epsilon, \quad (22)$$

where $D_{A_2B_2}=\sum_{i=0,1}|ii\rangle\langle ii|$ is the Choi-Jamiołkowski matrix of $\mathcal{I}_2$. Furthermore, $\rho_{A_1}\otimes\mathbb{1}_{B_1}-\widetilde{F}_{A_1B_1}\geq 0$ and consequently we find that $\rho_{A_1A_2}\otimes\mathbb{1}_{B_1B_2}-F_{A_1A_2B_1B_2}\geq 0$. Hence, $\left\{\frac{1}{2}\lambda,\rho_{A_1A_2},F_{A_1A_2B_1B_2}\right\}$ is a feasible solution, ensuring that

$$C^{(1)}_{\mathrm{NS}}(\mathcal{N}\otimes\mathcal{I}_2,\epsilon)-1\geq R(\mathcal{N},\epsilon). \quad (23)$$

■

Lemma 4 *We have* $R(\mathcal{N}\otimes\mathcal{I}_m,\epsilon)=R(\mathcal{N},\epsilon)+\log m$.

Proof: On one hand, it is easy to prove that $R(\mathcal{N}\otimes\mathcal{I}_m,\epsilon)\geq R(\mathcal{N},\epsilon)+\log m$. To see the other direction, we are going to use the dual SDP of $R(\mathcal{N},\epsilon)$:

$$\begin{aligned} R(\mathcal{N},\epsilon) = -\log\max\ & [s(1-\epsilon)-t] \\ \text{s.t. } & X_{AB}+\mathbb{1}_A\otimes Y_B\geq sJ_{\mathcal{N}}, \\ & \operatorname{Tr}_B X_{AB}\leq t\mathbb{1}_A, \operatorname{Tr}Y_B\leq 1, \\ & X_{AB},Y_B,s\geq 0. \end{aligned} \quad (24)$$

We note that the strong duality holds here.

Suppose that the optimal solution to the dual SDP (24) of $R(\mathcal{N},\epsilon)$ is $\{\widehat{X}_{AB},\widehat{Y}_B,\widehat{s},\widehat{t}\}$. Let us choose $X_{AA'BB'}=\frac{1}{m}\widehat{X}_{AB}\otimes D_m$, $Y_{BB'}=\frac{1}{m}\widehat{Y}_B\otimes\mathbb{1}_m$, $s=\frac{1}{m}\widehat{s}$, $t=\frac{1}{m}\widehat{t}$, with $D_m=\sum_{i=0}^{m-1}|ii\rangle\langle ii|$. Then it can be easily checked that

$$X_{AA'BB'}+\mathbb{1}_{AA'}\otimes Y_{BB'}\geq(\widehat{X}_{AB}+\mathbb{1}_A\otimes\widehat{Y}_B)\otimes\frac{D_m}{m} \quad (25)$$
$$\geq sJ_{\mathcal{N}}\otimes D_m. \quad (26)$$

The other constraints can be verified similarly. Thus, $\{X_{AA'BB'},Y_{BB'},s,t\}$ is a feasible solution to the SDP (24) of $R(\mathcal{N}\otimes\mathcal{I}_m,\epsilon)$, which means that

$$R(\mathcal{N}\otimes\mathcal{I}_m,\epsilon)\leq -\log[s(1-\epsilon)-t]=R(\mathcal{N},\epsilon)+\log m. \quad (27)$$

■

IV. New meta-converse for unassisted classical communication

Recall that the only useless quantum channel for classical communication is the *constant channel* $\mathcal{N}(\cdot)=\sigma$, which maps all states ρ on A to a constant state σ on B. As a natural extension, we say a subchannel $\mathcal{N}$ is *constant-bounded* if it maps all states ρ to positive definite operators that are smaller than or equal to a constant state σ, i.e.,

$$\mathcal{N}(\rho)\leq\sigma, \forall\rho\in\mathcal{S}(A). \quad (28)$$

Here we denote $\mathcal{S}(A):=\{\rho_A\geq 0:\operatorname{Tr}\rho_A=1\}$ as the set of quantum states on A, and a subchannel $\mathcal{N}$ is a linear completely positive (CP) map that is trace non-increasing, i.e., $\operatorname{Tr}\mathcal{N}(\rho)\leq 1$ for all states ρ.

We also define the set of *constant-bounded subchannels*:

$$\mathcal{V}:=\{\mathcal{M}\in\mathrm{CP}(A:B):\exists\,\sigma\in\mathcal{S}(B)\text{ s.t. }\mathcal{M}(\rho)\leq\sigma,\forall\rho\in\mathcal{S}(A)\},$$

where $\mathrm{CP}(A:B)$ denotes the set of all CP maps from A to B. Clearly, $\mathcal{V}$ is convex and closed.

This inspires the following new one-shot converse bound:

Theorem 5 *For any quantum channel* $\mathcal{N}_{A'\to B}$ *and error tolerance* $\epsilon\in(0,1)$*, we have*

$$C^{(1)}(\mathcal{N},\epsilon)\leq\max_{\rho_{A'}}\min_{\mathcal{M}\in\mathcal{V}}D_H^{\epsilon}(\mathcal{N}_{A'\to B}(\phi_{A'A})\|\mathcal{M}_{A'\to B}(\phi_{A'A})) \quad (29)$$
$$=\min_{\mathcal{M}\in\mathcal{V}}\max_{\rho_{A'}}D_H^{\epsilon}(\mathcal{N}_{A'\to B}(\phi_{A'A})\|\mathcal{M}_{A'\to B}(\phi_{A'A})), \quad (30)$$

where $\phi_{A'A}$ *is a purification of* $\rho_{A'}$.

Proof: Consider an unassisted code with inputs $\{\rho_k\}_{k=1}^m$ and POVM $\{M_k\}_{k=1}^m$ whose average input state is $\rho_{A'} = \sum_{k=1}^m \frac{1}{m}\rho_k$, the success probability to transmit m messages is given by

$$p_{\text{succ}} = \frac{1}{m}\sum_{k=1}^m \operatorname{Tr}\mathcal{N}(\rho_k)M_k \tag{31}$$

$$= \operatorname{Tr} J_{\mathcal{N}}(\sum_{k=1}^m \frac{1}{m}\rho_k^T \otimes M_k) \tag{32}$$

$$= \operatorname{Tr}\mathcal{N}_{A'\to B}(\phi_{AA'})(\rho_A^T)^{-1/2}(\sum_{k=1}^m \frac{1}{m}\rho_k^T \otimes M_k)(\rho_A^T)^{-1/2}. \tag{33}$$

Denote $E = (\rho_A^T)^{-1/2}(\sum_{k=1}^m \frac{1}{m}\rho_k^T \otimes M_k)(\rho_A^T)^{-1/2}$. Then

$$0 \le E \le (\rho_A^T)^{-1/2}(\sum_{k=1}^m \frac{1}{m}\rho_k^T \otimes \mathbb{1}_B)(\rho_A^T)^{-1/2} = \mathbb{1}_{AB}. \tag{34}$$

For any $\mathcal{M} \in \mathcal{V}$, we assume that the output states of $\mathcal{M}$ are bounded by the state σ_B, then

$$\operatorname{Tr}\mathcal{M}_{A'\to B}(\phi_{AA'})E \tag{35}$$

$$= \operatorname{Tr}\mathcal{M}_{A'\to B}(\phi_{AA'})(\rho_A^T)^{-1/2}(\sum_{k=1}^m \frac{1}{m}\rho_k^T \otimes M_k)(\rho_A^T)^{-1/2} \tag{36}$$

$$= \operatorname{Tr} J_{\mathcal{M}}(\sum_{k=1}^m \frac{1}{m}\rho_k^T \otimes M_k) \tag{37}$$

$$= \frac{1}{m}\sum_{k=1}^m \operatorname{Tr}\mathcal{M}(\rho_k)M_k \le \frac{1}{m}\sum_{k=1}^m \operatorname{Tr}\sigma_B M_k = \frac{1}{m}. \tag{38}$$

The second line follows from the fact that $J_{\mathcal{M}} = (\rho_A^T)^{-1/2}\mathcal{M}_{A'\to B}(\phi_{AA'})(\rho_A^T)^{-1/2}$. In the third line, we use the inverse Choi-Jamiołkowski transformation $\mathcal{M}_{A'\to B}(\rho_{A'}) = \operatorname{Tr}_A J_{\mathcal{M}}(\rho_A^T \otimes \mathbb{1}_B)$. The forth line follows since any output state of $\mathcal{M}$ is bounded by the state σ_B.

Therefore, combining Eqs. (33) and (38), we know that $\operatorname{Tr}\mathcal{N}_{A'\to B}(\phi_{AA'})E \ge 1-\varepsilon$ and $\operatorname{Tr}\mathcal{M}_{A'\to B}(\phi_{AA'})E \le \frac{1}{m}$. Thus $C^{(1)}(\mathcal{N},\rho_{A'},\varepsilon) \le \min_{\mathcal{M}\in\mathcal{V}} D_H^\varepsilon(\mathcal{N}_{A'\to B}(\phi_{AA'})\|\mathcal{M}_{A'\to B}(\phi_{AA'}))$. Maximizing over all average input $\rho_{A'}$, we can obtain the desired result of (29).

Since $\beta_\varepsilon(\mathcal{N}_{A'\to B}(\phi_{A'A})\|\mathcal{M}_{A'\to B}(\phi_{A'A}))$ is convex in $\rho_{A'}$ and concave in $\mathcal{M}$ [16], we can exchange the maximization and minimization by applying Sion's minimax theorem [39] and obtain the result of (30).

We note that the operator E above also satisfies $0 \le E^{T_B} \le \mathbb{1}$, where T_B means the partial transpose on system B. Then we can further add the PPT constraint on E. ■

If we consider $\max_{\rho_{A'}} D_H^\epsilon(\mathcal{N}_{A'\to B}(\phi_{A'A})\|\mathcal{M}_{A'\to B}(\phi_{A'A}))$ as the "distance" between the channel $\mathcal{N}$ and CP map $\mathcal{M}$. Then our new meta-converse can be treated as the "distance" between the given channel $\mathcal{N}$ with the class of useless constant-bounded subchannels.

We then restrict the set of constant-bounded subchannels $\mathcal{V}$ to an SDP-tractable set of CP maps. Let us define

$$\mathcal{V}_\beta := \{\mathcal{M} \in \text{CP}(A:B) : \beta(J_{\mathcal{M}}) \le 1\}, \quad \text{where} \tag{39}$$

$$\begin{aligned}\beta(J_{\mathcal{M}}) &:= \min \operatorname{Tr} S_B \\ &\text{s.t. } -R_{AB} \le J_{\mathcal{M}}^{T_B} \le R_{AB}, \\ &\quad -\mathbb{1}_A \otimes S_B \le R_{AB}^{T_B} \le \mathbb{1}_A \otimes S_B.\end{aligned} \tag{40}$$

Here $J_{\mathcal{M}}$ is the Choi-Jamiołkowski matrix of $\mathcal{M}$. Note that $\log\beta(\mathcal{N})$ is an SDP strong converse bound on the classical capacity of $\mathcal{N}$ established in [17].

The set $\mathcal{V}_\beta$ is actually a subset of $\mathcal{V}$, i.e., $\mathcal{V}_\beta \subset \mathcal{V}$. Then the set $\mathcal{V}$ in Theorem 5 can replaced by $\mathcal{V}_\beta$ and one could get an SDP-computable upper bound on the one-shot ε-error capacity. (See [40] for the details.)

There are several other converses for the one-shot ε-error capacity of a general quantum channel, e.g., the Matthews-Wehner converse [16], the Datta-Hsieh converse [41], and the recent SDP converse via no-signaling (NS) and positive-partial-transpose-preserving (PPT) codes [17]. Note that the Datta-Hsieh converse is not known to be efficiently computable. Also, our meta-converse is tighter than the Matthews-Wehner converse in Eq. (5), but it is no better than the SDP converse via NS and PPT codes [17].

As an application, we apply our meta-converse to establish second-order asymptotics [21] of the quantum erasure channel.

Theorem 6 *For any quantum erasure channel $\mathcal{E}_p$ with parameter p and input dimension d, we have*

$$\begin{aligned}&C^{(1)}(\mathcal{E}_p^{\otimes n},\varepsilon) \\ =&n(1-p)\log d + \sqrt{np(1-p)(\log d)^2}\ \Phi^{-1}(\varepsilon) + O(\log n),\end{aligned} \tag{41}$$

where Φ is the cumulative distribution function of a standard normal random variable.

Proof: For the direct part, denote $\mathcal{F}_1(\rho) = \sum_{i=0}^{d-1}\langle i|\rho|i\rangle|i\rangle\langle i|$, and $\mathcal{F}_2(\rho) = \sum_{i=0}^{d}\langle i|\rho|i\rangle|i\rangle\langle i|$, which are both classical channels. Then $\mathcal{N}_p = \mathcal{F}_2 \circ \mathcal{E}_p \circ \mathcal{F}_1$ is a classical erasure channel. Then one can apply the result in [19] to get the lower bound.

For the converse part, we have

$$C^{(1)}(\mathcal{E}_p^{\otimes n},\varepsilon) \le \min_{\mathcal{M}\in\mathcal{V}} D_H^\varepsilon(\mathcal{E}_p^{\otimes n}(\Phi_{A'^nA^n})\|\mathcal{M}_{A'^n\to B^n}(\Phi_{A'^nA^n})). \tag{42}$$

Let us denote

$$J_{\mathcal{M}} = \frac{1-p}{d}\sum_{i,j=0}^{d-1}|ii\rangle\langle jj| + p\sum_{i=0}^{d-1}|i\rangle\langle i| \otimes |d\rangle\langle d| \tag{43}$$

as the Choi-Jamiołkowski matrix of the CP map $\mathcal{M}$.

Take $\mathcal{M}_{A'^n\to B^n} = \mathcal{M}_{A'\to B}^{\otimes n}$, we have

$$\begin{aligned}&D_H^\varepsilon(\mathcal{E}_p^{\otimes n}(\Phi_{A'^nA^n})\|\mathcal{M}_{A'\to B}^{\otimes n}(\Phi_{A'^nA^n})) \\ =&nD(\mathcal{E}_p(\Phi_{A'A})\|\mathcal{M}(\Phi_{A'A})) \\ &+ \sqrt{nV(\mathcal{E}_p(\Phi_{A'A})\|\mathcal{M}(\Phi_{A'A}))}\ \Phi^{-1}(\varepsilon) + O(\log n) \quad (44) \\ =&n(1-p)\log d + \sqrt{np(1-p)(\log d)^2}\ \Phi^{-1}(\varepsilon) + O(\log n).\end{aligned} \tag{45}$$

In the second line, we use second-order expansion of quantum hypothesis testing relative entropy [42], [43]. The third line follows by direct calculation. Combining this with (42) leads to the desired bound. ■

ACKNOWLEDGEMENT

We were grateful to Mark M. Wilde for comments on a previous version of this manuscript which inspired us to improve Theorem 5. XW and KF were partly supported by the Australian Research Council under Grant Nos. DP120103776 and FT120100449. MT acknowledges an Australian Research Council Discovery Early Career Researcher Award, project No. DE160100821.

REFERENCES

[1] A. S. Holevo, "Bounds for the quantity of information transmitted by a quantum communication channel," *Problemy Peredachi Informatsii (Problems of Information Transmission)*, vol. 9, no. 3, pp. 3–11, 1973.

[2] A. Holevo, "The capacity of the quantum channel with general signal states," *IEEE Transactions on Information Theory*, vol. 44, no. 1, pp. 269–273, 1998.

[3] B. Schumacher and M. D. Westmoreland, "Sending classical information via noisy quantum channels," *Physical Review A*, vol. 56, no. 1, pp. 131–138, jul 1997.

[4] C. King, "The capacity of the quantum depolarizing channel," *IEEE Transactions on Information Theory*, vol. 49, no. 1, pp. 221–229, jan 2003.

[5] C. H. Bennett, D. P. DiVincenzo, and J. A. Smolin, "Capacities of Quantum Erasure Channels," *Physical Review Letters*, vol. 78, no. 16, pp. 3217–3220, apr 1997.

[6] C. King, "Additivity for unital qubit channels," *Journal of Mathematical Physics*, vol. 43, no. 10, p. 4641, 2002.

[7] G. G. Amosov, A. S. Holevo, and R. F. Werner, "On Some Additivity Problems in Quantum Information Theory," *Problemy Peredachi Informatsii*, vol. 36, no. 4, pp. 25–34, 2000.

[8] N. DATTA, A. S. HOLEVO, and Y. SUHOV, "Additivity for transpose depolarizing channels," *International Journal of Quantum Information*, vol. 04, no. 01, pp. 85–98, feb 2006.

[9] M. Fukuda, "Extending additivity from symmetric to asymmetric channels," *Journal of Physics A: Mathematical and General*, vol. 38, no. 45, pp. L753–L758, nov 2005.

[10] R. Konig, S. Wehner, and J. Wullschleger, "Unconditional Security From Noisy Quantum Storage," *IEEE Transactions on Information Theory*, vol. 58, no. 3, pp. 1962–1984, mar 2012.

[11] M. B. Hastings, "Superadditivity of communication capacity using entangled inputs," *Nature Physics*, vol. 5, no. 4, pp. 255–257, apr 2009.

[12] M. Mosonyi and N. Datta, "Generalized relative entropies and the capacity of classical-quantum channels," *Journal of Mathematical Physics*, vol. 50, no. 7, p. 072104, jul 2009.

[13] L. Wang and R. Renner, "One-Shot Classical-Quantum Capacity and Hypothesis Testing," *Physical Review Letters*, vol. 108, no. 20, p. 200501, may 2012.

[14] J. M. Renes and R. Renner, "Noisy Channel Coding via Privacy Amplification and Information Reconciliation," *IEEE Transactions on Information Theory*, vol. 57, no. 11, pp. 7377–7385, nov 2011.

[15] M. Hayashi, *Quantum Information*. Springer, 2006, no. 1.

[16] W. Matthews and S. Wehner, "Finite Blocklength Converse Bounds for Quantum Channels," *IEEE Transactions on Information Theory*, vol. 60, no. 11, pp. 7317–7329, nov 2014.

[17] X. Wang, W. Xie, and R. Duan, "Semidefinite programming strong converse bounds for classical capacity," *IEEE Transactions on Information Theory*, vol. 64, no. 1, pp. 640–653, oct 2018.

[18] W. Matthews, "A Linear Program for the Finite Block Length Converse of PolyanskiyPoorVerdú Via Nonsignaling Codes," *IEEE Transactions on Information Theory*, vol. 58, no. 12, pp. 7036–7044, dec 2012.

[19] Y. Polyanskiy, H. V. Poor, and S. Verdu, "Channel Coding Rate in the Finite Blocklength Regime," *IEEE Transactions on Information Theory*, vol. 56, no. 5, pp. 2307–2359, may 2010.

[20] H. Nagaoka, "Strong converse theorems in quantum information theory," in *ERATO Workshop on Quantum Information Science 2001*, Tokyo, 2001, p. 33.

[21] M. Tomamichel and V. Y. F. Tan, "Second-Order Asymptotics for the Classical Capacity of Image-Additive Quantum Channels," *Communications in Mathematical Physics*, vol. 338, no. 1, pp. 103–137, aug 2015.

[22] D. Leung and W. Matthews, "On the Power of PPT-Preserving and Non-Signalling Codes," *IEEE Transactions on Information Theory*, vol. 61, no. 8, pp. 4486–4499, aug 2015.

[23] R. Duan and A. Winter, "No-Signalling-Assisted Zero-Error Capacity of Quantum Channels and an Information Theoretic Interpretation of the Lovász Number," *IEEE Transactions on Information Theory*, vol. 62, no. 2, pp. 891–914, feb 2016.

[24] X. Wang and R. Duan, "A semidefinite programming upper bound of quantum capacity," in *2016 IEEE International Symposium on Information Theory (ISIT)*, vol. 2016-Augus. IEEE, jul 2016, pp. 1690–1694.

[25] T. S. Cubitt, D. Leung, W. Matthews, and A. Winter, "Zero-Error Channel Capacity and Simulation Assisted by Non-Local Correlations," *IEEE Transactions on Information Theory*, vol. 57, no. 8, pp. 5509–5523, aug 2011.

[26] X. Wang and R. Duan, "Separation Between Quantum Lovász Number and Entanglement-Assisted Zero-Error Classical Capacity," *IEEE Transactions on Information Theory*, vol. 64, no. 3, pp. 1454–1460, mar 2018.

[27] J. Park and S. Lee, "Quantum nonsignaling-assisted zero-error classical capacity of qubit channels," *Physical Review A*, vol. 93, no. 3, p. 032334, mar 2016.

[28] X. Wang and R. Duan, "On the quantum no-signalling assisted zero-error classical simulation cost of non-commutative bipartite graphs," in *2016 IEEE International Symposium on Information Theory (ISIT)*, vol. 2016-Augus. IEEE, jul 2016, pp. 2244–2248.

[29] W. Xie, X. Wang, and R. Duan, "Converse bounds for classical communication over quantum networks," *arXiv:1712.05637*, jun 2017.

[30] C. W. Helstrom, *Quantum detection and estimation theory*. Academic press, 1976, vol. 123.

[31] F. Hiai and D. Petz, "The proper formula for relative entropy and its asymptotics in quantum probability," *Communications in Mathematical Physics*, vol. 143, no. 1, pp. 99–114, dec 1991.

[32] H. Nagaoka and T. Ogawa, "Strong converse and Stein's lemma in quantum hypothesis testing," *IEEE Transactions on Information Theory*, vol. 46, no. 7, pp. 2428–2433, feb 2000.

[33] M.-D. Choi, "Completely positive linear maps on complex matrices," *Linear Algebra and its Applications*, vol. 10, no. 3, pp. 285–290, jun 1975.

[34] A. Jamiołkowski, "Linear transformations which preserve trace and positive semidefiniteness of operators," *Reports on Mathematical Physics*, vol. 3, no. 4, pp. 275–278, dec 1972.

[35] L. Vandenberghe and S. Boyd, "Semidefinite Programming," *SIAM Review*, vol. 38, no. 1, pp. 49–95, mar 1996.

[36] A. Winter and D. Yang, "Potential capacities of quantum channels," *IEEE Transactions on Information Theory*, vol. 62, no. 3, pp. 1415–1424, may 2015.

[37] A. Acín, R. Duan, D. E. Roberson, A. B. Sainz, and A. Winter, "A new property of the Lovász number and duality relations between graph parameters," *Discrete Applied Mathematics*, vol. 216, pp. 489–501, jan 2017.

[38] R. Duan and X. Wang, "Activated zero-error classical communication over quantum channels assisted with quantum no-signalling correlations," *arXiv:1510.05437*, oct 2015.

[39] M. Sion, "On general minimax theorems," *Pacific Journal of Mathematics*, vol. 8, no. 1, pp. 171–176, mar 1958.

[40] X. Wang, K. Fang, and M. Tomamichel, "On converse bounds for classical communication over quantum channels," *arXiv:1709.05258*, 2017.

[41] N. Datta and M.-H. Hsieh, "One-Shot Entanglement-Assisted Quantum and Classical Communication," *IEEE Transactions on Information Theory*, vol. 59, no. 3, pp. 1929–1939, mar 2013.

[42] M. Tomamichel and M. Hayashi, "A Hierarchy of Information Quantities for Finite Block Length Analysis of Quantum Tasks," *IEEE Transactions on Information Theory*, vol. 59, no. 11, pp. 7693–7710, nov 2013.

[43] K. Li, "Second-order asymptotics for quantum hypothesis testing," *The Annals of Statistics*, vol. 42, no. 1, pp. 171–189, feb 2014.