Efficiently computable upper bounds for quantum communication

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Based on the joint submission

(1709.00200)

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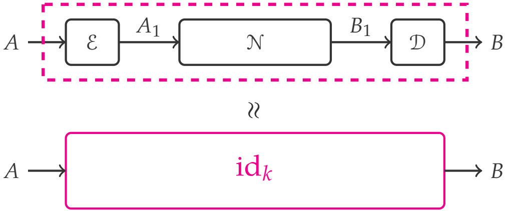

How good the simulation is? [Kretschmann, Werner, 2004]

- Channel distance $\|\mathrm{id}_k - \mathcal{D} \circ \mathcal{N} \circ \mathcal{E}\|_\diamond$.
- Channel fidelity $F(\Phi_k, \mathcal{D} \circ \mathcal{N} \circ \mathcal{E}(\Phi_k))$, where $|\Phi_k\rangle = \frac{1}{\sqrt{k}} \sum_{i=1}^k |ii\rangle$. ✓
- ...

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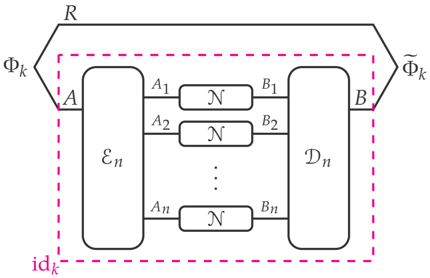


- r: qubits transmitted per channel use.
- n: number of channel uses.
- ε: error tolerance.
- (r, n, ε) achievable: exists Φ_k, $\mathcal{E}_n$ and $\mathcal{D}_n$

  s.t. $r = \frac{1}{n}\log_2 k$, $F\left(\Phi_k, \widetilde{\Phi}_k\right) \geq 1 - \varepsilon$.

- Quantum capacity

$$Q(\mathcal{N}) := \lim_{\varepsilon \to 0} \lim_{n \to \infty} \sup\{r : (r, n, \varepsilon) \text{ achievable}\}.$$

- Strong converse rate r_0:

  for any achievable (r, n, ε) such that $r \geq r_0$, then $\varepsilon \to 1$ as $n \to \infty$.

- Strong converse quantum capacity

$$Q^{\dagger}(\mathcal{N}) := \inf\{r_0 : r_0 \text{ strong converse rate}\}.$$

- For any quantum channel $\mathcal{N}$, it holds $Q(\mathcal{N}) \leq Q^{\dagger}(\mathcal{N})$.

Theorem (Barnum, Nielsen, Schumacher, 1996-2000; Lloyd, Shor, Devetak, 1997-2005)

For any quantum channel $\mathcal{N}$, its quantum capacity is equal to the regularized coherent information of the channel:

$$Q(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} I_c\left(\mathcal{N}^{\otimes n}\right),$$

where $I_c(\mathcal{N}) = \max_{\phi_{AA'}} I(A\rangle B)_{\mathcal{N}_{A'\to B}(\phi_{AA'})}$ *and* $\phi_{AA'}$ *is a pure state.*

- Not a single-letter formula.
- $n \to \infty$ is necessary in general [Cubitt el.al, 2014].
- $I_c(\mathcal{N})$ not additive in general.
- $Q(\mathcal{N})$ not additive in general [Smith, Yard, 2009]

Difficult to compute!

Even for qubit depolarizing channel

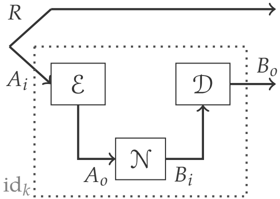
$$\mathcal{N}(\rho) = (1-p)\,\rho + p\frac{\mathbb{1}}{2},$$

its quantum capacity is still unknown.
For most recent result, refer to [Sutter et.al, 2014; Leditzky et.al, 2017]

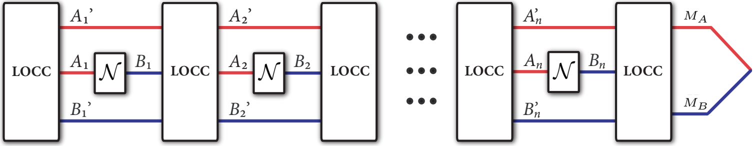
Some known converse (upper) bounds:

- R: Rains information [Tomamichel et.al, 2014]
- ε-DEG: Epsilon degradable bound [Sutter et.al, 2014]
- E_{sq}: Squashed entanglement of a channel [Takeoka et.al, 2013]
- E_C: Entanglement cost of a channel [Berta et.al, 2011]
- Q_E: Entanglement assisted quantum capacity [Bennett et.al, 2009]
- Q_{ss}: Quantum capacity with symmetric side channels [Smith et.al, 2006]
- Q_Θ: Partial transposition bound [Holevo,Werner, 1999; Muller-Hermes et.al, 2015]

Have a summary later...



Result 1: improved SDP one-shot converse bound.



Result 2: improved SDP strong converse bound for LOCC-assisted quantum capacity.

Converse bounds for one-shot quantum capacity

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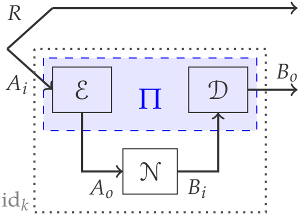
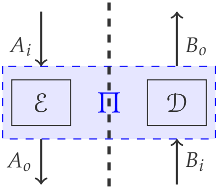


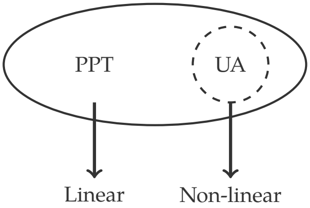


- Unassisted code (UA):

$$\Pi_{A_i B_i \to A_o B_o} = \mathcal{E}_{A_i \to A_o} \otimes \mathcal{D}_{B_i \to B_o}.$$

- Positive-partial-transpose (PPT) code: [Rains, 1999 & 2001]

$$J_\Pi = \Pi_{A_i B_i \to A_o B_o}\left(\Phi_{A_i B_i : A_i' B_i'}\right), \quad J_\Pi^{T_{B_i B_o}} \geq 0.$$



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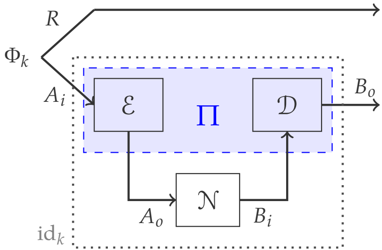

Maximum channel fidelity

$$F_\Omega(\mathcal{N},k) := \sup_{\Pi\in\Omega} F(\underbrace{\Phi_k}_{\text{input}}, \underbrace{\Pi\circ\mathcal{N}(\Phi_k)}_{\text{output}}).$$

where $\Omega \in \{\mathrm{UA}, \mathrm{PPT}\}$.

One-shot quantum capacity

error tolerance

$$Q^{(1)}_\Omega(\mathcal{N},\varepsilon) := \log\max\{k : F_\Omega(\mathcal{N},k) \geq 1-\varepsilon\}.$$

(Asymptotic) quantum capacity

$$Q_\Omega(\mathcal{N}) = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}Q^{(1)}_\Omega(\mathcal{N}^{\otimes n},\varepsilon).$$

[Tomamichel, Berta, Renes, 2016] $Q^{(1)}(\mathcal{N}, \varepsilon) \leq -\log f(\mathcal{N}, \varepsilon)$,

$$\begin{aligned} \text{where } f(\mathcal{N}, \varepsilon) = \min \operatorname{Tr} S_A \\ \text{s.t. } \operatorname{Tr} J_{\mathcal{N}} W_{AB} \geq 1-\varepsilon, S_A \geq 0, \Theta_{AB} \geq 0, \operatorname{Tr} \rho_A = 1, \\ 0 \leq W_{AB} \leq \rho_A \otimes \mathbb{1}_B, S_A \otimes \mathbb{1}_B \geq W_{AB} + \Theta_{AB}^{T_B}. \end{aligned} \quad (1)$$

SDP: linear objective function with semidefintie conditions.

Main Result 1: improved SDP converse for one-shot capacity

For any quantum channel $\mathcal{N}$ and error tolerance ε, the inequality chain holds

$$Q^{(1)}(\mathcal{N}, \varepsilon) \leq Q^{(1)}_{PPT}(\mathcal{N}, \varepsilon) \leq -\log g(\mathcal{N}, \varepsilon) \leq -\log f(\mathcal{N}, \varepsilon). \quad (2)$$

- **Step 1:** Derive the optimization for $Q^{(1)}_{PPT}(\mathcal{N}, \varepsilon)$;
- **Step 2:** Relax $Q^{(1)}_{PPT}(\mathcal{N}, \varepsilon)$ to a semidefinite program $-\log g(\mathcal{N}, \varepsilon)$;
- **Step 3:** Prove $-\log g(\mathcal{N}, \varepsilon) \leq -\log f(\mathcal{N}, \varepsilon)$.

[Leung, Matthews, 2015]

$$F_{PPT}(\mathcal{N},k) = \max \operatorname{Tr} J_{\mathcal{N}} W_{AB} \text{ s.t. } 0 \le W_{AB} \le \rho_A \otimes \mathbb{1}_B, \operatorname{Tr}\rho_A = 1,$$
$$-k^{-1}\rho_A \otimes \mathbb{1}_B \le W_{AB}^{T_B} \le k^{-1}\rho_A \otimes \mathbb{1}_B.$$

Use the definition that $Q^{(1)}_{PPT}(\mathcal{N},\varepsilon) := \log\max\{k : F_{PPT}(\mathcal{N},k) \ge 1-\varepsilon\}$,

Step 1: $Q^{(1)}_{PPT}(\mathcal{N},\varepsilon) = -\log\min m$

$$\text{s.t. } \operatorname{Tr} J_{\mathcal{N}} W_{AB} \ge 1-\varepsilon, 0 \le W_{AB} \le \rho_A \otimes \mathbb{1}_B,$$
$$\operatorname{Tr}\rho_A = 1, -m\rho_A \otimes \mathbb{1}_B \le W_{AB}^{T_B} \le m\rho_A \otimes \mathbb{1}_B.$$

Step 2: $-\log g(\mathcal{N},\varepsilon) := -\log\min \operatorname{Tr} S_A$

$$\text{s.t. } \operatorname{Tr} J_{\mathcal{N}} W_{AB} \ge 1-\varepsilon, 0 \le W_{AB} \le \rho_A \otimes \mathbb{1}_B,$$
$$\operatorname{Tr}\rho_A = 1, -S_A \otimes \mathbb{1}_B \le W_{AB}^{T_B} \le S_A \otimes \mathbb{1}_B.$$

Thus $Q^{(1)}(\mathcal{N},\varepsilon) \le Q^{(1)}_{PPT}(\mathcal{N},\varepsilon) \le -\log g(\mathcal{N},\varepsilon)$.

Step 3: Prove $-\log g(\mathcal{N},\varepsilon) \le -\log f(\mathcal{N},\varepsilon)$ by constructing feasible solutions.

Example: $-\log g\,(\mathcal{N}, \varepsilon) < -\log f\,(\mathcal{N}, \varepsilon)$.

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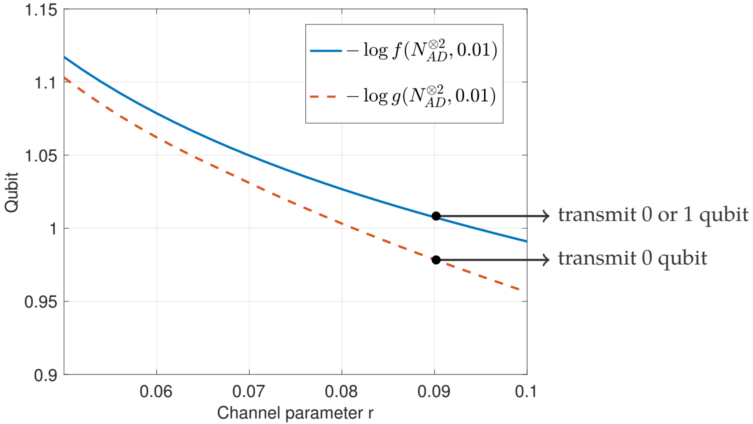
Amplitude damping channel $\mathcal{N}_{AD} = \sum_{i=0}^{1} E_i \cdot E_i^\dagger$ with

$$E_0 = |0\rangle\langle 0| + \sqrt{1-r}|1\rangle\langle 1| \quad E_1 = \sqrt{r}|0\rangle\langle 1|, \quad 0 \le r \le 1.$$



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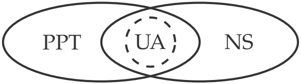
We can further improve the SDP converse bound by considering non-singalling codes.



Main Result 1: improved SDP converse for one-shot capacity

For any quantum channel $\mathcal{N}$ and error tolerance ε, the inequality chain holds

$$\begin{aligned} Q^{(1)}(\mathcal{N},\varepsilon) &\leq Q^{(1)}_{PPT\cap NS}(\mathcal{N},\varepsilon) \\ &\leq -\log \widetilde{g}(\mathcal{N},\varepsilon) \leq -\log \widehat{g}(\mathcal{N},\varepsilon) \leq -\log g(\mathcal{N},\varepsilon) \leq -\log f(\mathcal{N},\varepsilon). \end{aligned} \tag{3}$$

Converse bound for asymptotic quantum capacity

[Wang, Duan, 2016] introduced an SDP converse bound $Q_\Gamma(\mathcal{N})$ for quantum capacity, i.e., $Q(\mathcal{N}) \le Q_\Gamma(\mathcal{N})$, where

$$\begin{aligned} Q_\Gamma(\mathcal{N}) := \log \max \operatorname{Tr} J_{\mathcal{N}} R_{AB} \\ \text{s.t. } R_{AB} \ge 0, \rho_A \ge 0, \operatorname{Tr} \rho_A = 1, \\ -\rho_A \otimes \mathbb{1}_B \le R_{AB}{}^{T_B} \le \rho_A \otimes \mathbb{1}_B. \end{aligned} \tag{4}$$

Some nice properties:

- Additivity: $Q_\Gamma(\mathcal{N} \otimes \mathcal{M}) = Q_\Gamma(\mathcal{N}) + Q_\Gamma(\mathcal{M})$ (by utilizing SDP duality).
- For noiseless quantum channel id_m, $Q(\mathrm{id}_m) = Q_\Gamma(\mathrm{id}_m) = \log_2 m$.
- Strong converse: achievable (r, n, ε) satisfies $\varepsilon \ge 1 - 2^{-n(r - Q_\Gamma(\mathcal{N}))}$.
- Tighter than the Partial Transposition bound [Holevo, Werner, 2001], i.e.,

$$Q_\Gamma(\mathcal{N}) \le Q_\Theta(\mathcal{N}) := \log \|T \circ \mathcal{N}\|_\diamond,$$

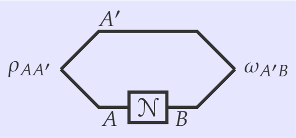
  where T is the transpose map, $\|\cdot\|_\diamond$ is the diamond norm [Aharonov et.al, 1998].

We have a better understanding of $Q_\Gamma(\mathcal{N})$ now.

For any vector norm $\|\cdot\|$, we can define its induced norm $\|A\| = \sup_{\|x\|=1} \|Ax\|$.

Similarly, given an entanglement measure E, the **entanglement of a quantum channel** is defined as

$$E(\mathcal{N}) := \sup_{\rho_{A'A}} E(A':B)_\omega\,, \text{ where } \omega_{A'B} = \mathcal{N}_{A\to B}(\rho_{A'A})\,.$$



Consider the entanglement measure $R_{\max}$ in [Wang, Duan, 2016]

$$R_{\max}(\rho) := \log \max \left\{ \operatorname{Tr} \rho R_{AB} : -\mathbb{1}_{AB} \le R_{AB}^{T_B} \le \mathbb{1}_{AB}, R_{AB} \ge 0 \right\}, \tag{5}$$

$$= \min_{\sigma \in \mathrm{PPT}'} D_{\max}(\rho\|\sigma), \quad \left[\text{Rains bound: } R(\rho) = \min_{\sigma\in\mathrm{PPT}'} D(\rho\|\sigma)\right] \tag{6}$$

where the Rains set $\mathrm{PPT}' := \{\sigma \ge 0 : \|\sigma^{T_B}\|_1 \le 1\}$ and $D_{\max}(\rho\|\sigma) := \log \inf\{t : \rho \le t\sigma\}$. Then we have

$$Q_\Gamma(\mathcal{N}) = \max_{\rho_{AA'}} R_{\max}(\mathcal{N}_{A\to B}(\rho_{AA'}))\,. \tag{7}$$

Rains bound [Rains, 2001]		$R_{\max}$ [Wang, Duan, 2016]		Log-negativity [Vidal, Werner, 2001]
$R(\rho) = \min_{\sigma \in \text{PPT}'} D(\rho\|\sigma)$ [Audenaert et.al, 2001]	$\leq$	$R_{\max}(\rho) = \min_{\sigma \in \text{PPT}'} D_{\max}(\rho\|\sigma)$ [Wang et.al, 2017]	$\leq$	$E_N(\rho) = \log \|\rho^{T_B}\|_1$
Rains information [Tomamichel et.al, 2014]		$Q_\Gamma(R_{\max})$ [Wang, Duan,2016]		Partial trans. bound [Holevo, Werner, 2001]
$R(\mathcal{N}) = \sup_{\rho_{A'A}} R(\omega)$	$\leq$	$Q_\Gamma(\mathcal{N}) = \sup_{\rho_{A'A}} R_{\max}(\omega)$	$\leq$	$Q_\Theta(\mathcal{N}) = \sup_{\rho_{A'A}} E_N(\omega)$

Thus it is clear that

$$Q(\mathcal{N}) \leq Q^\dagger(\mathcal{N}) \leq R(\mathcal{N}) \leq \underset{\overset{||}{Q_\Gamma(\mathcal{N})}}{R_{\max}(\mathcal{N})} \leq Q_\Theta(\mathcal{N}).$$

- $R(\mathcal{N})$ is a strong converse but not known to be efficiently computable in general.
- $R_{\max}(\mathcal{N})$ is a strong converse and **efficiently computable** in general.

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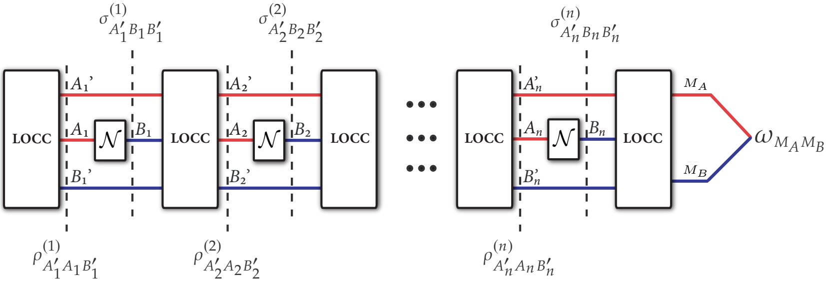
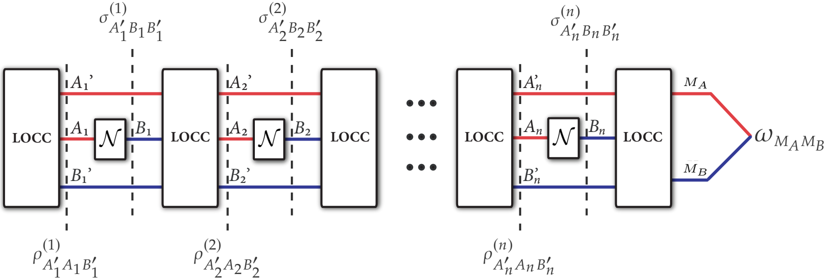


LOCC: local operations and classical communication.

The most relevant setting in practice

but much more complicated due to the potentially infinite rounds of c.c.

LOCC-assisted quantum capacity



(r, n, ε) is achievable if $\exists$ $\{\mathrm{LOCC}_n\}$, such that $r = \frac{1}{n}\log_2 k$ and $F\left(\omega_{M_A M_B}, \Phi_k\right) \geq 1 - \varepsilon$.

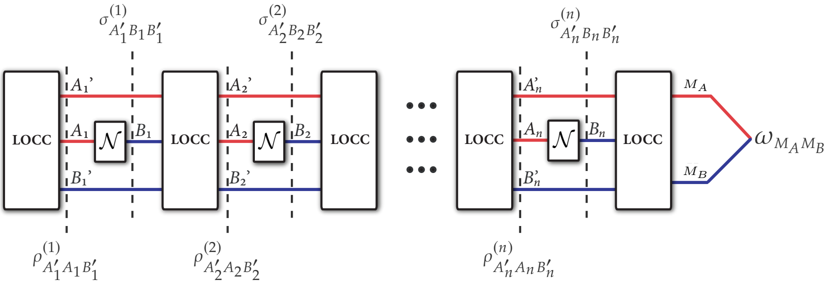
$$Q^{\leftrightarrow}(\mathcal{N}) := \lim_{\varepsilon \to 0} \lim_{n \to \infty} \sup\{r : (r, n, \varepsilon) \text{ achievable}\}.$$

Strong converse rate r_0:

for any achievable (r, n, ε) such that $r \geq r_0$, then $\varepsilon \to 1$ as $n \to \infty$.

Strong converse LOCC-assisted quantum capacity

$$Q^{\leftrightarrow,\dagger}(\mathcal{N}) := \inf\{r_0 : r_0 \text{ strong converse rate}\}.$$



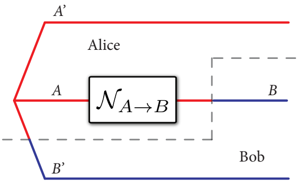
Main Result 2: improved SDP strong converse for $Q^{\leftrightarrow}$

For any quantum channel $\mathcal{N}$, it holds $Q^{\leftrightarrow}(\mathcal{N}) \leq Q^{\leftrightarrow,\dagger}(\mathcal{N}) \leq R_{\max}(\mathcal{N}) \leq Q_{\Theta}(\mathcal{N})$.

This established the tightest known efficiently computable strong converse bound on LOCC-assisted quantum capacity of an arbitrary channel.

Note: See also the strong converse bound for the LOCC-assisted private capacity in

[Christandl, Müller-Hermes, 2016].



Quantum channel $\mathcal{N}_{A\to B}$, entanglement measure E, Define the **amortized entanglement of the channel** as follows:

$$E_A(\mathcal{N}) := \sup_{\rho_{A'AB'}} \underbrace{E(A':BB')_\omega - E(A'A:B')_\rho}_{\text{net amount of ent.}},$$

where $\omega_{A'BB'} = \mathcal{N}_{A\to B}(\rho_{A'AB})$.

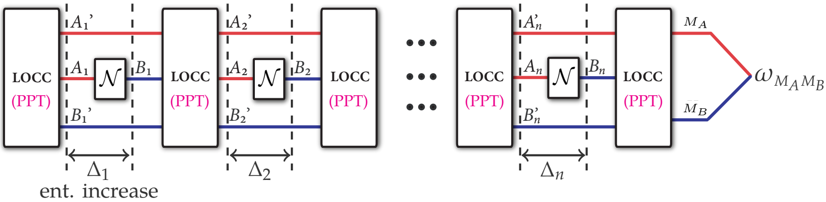
Recall that the entanglement of the channel is

$E(\mathcal{N}) := \sup_{\rho_{A'A}} E(A':B)_\omega$, where $\omega_{A'B} = \mathcal{N}_{A\to B}(\rho_{A'A})$.

- It is clear that $E(\mathcal{N}) \le E_A(\mathcal{N})$ since we could take the B' system trivial in $E_A(\mathcal{N})$.
- For squashed entanglement, $E_{sq}(\mathcal{N}) = E_{sq,A}(\mathcal{N})$ [Takeoka et.al,2014].
- For max-Rains information, $R_{\max}(\mathcal{N}) = R_{\max,A}(\mathcal{N})$.

We only need to prove that $R_{\max}(A':BB')_\omega - R_{\max}(A'A:B)_\rho \le R_{\max}(\mathcal{N})$.

All terms are SDPs and the inequality can be shown by constructing feasible solutions.



Since $R_{\max}(\mathcal{E}(\rho)) \le R_{\max}(\rho)$ for any PPT operation $\mathcal{E}$, we have

$$R_{\max}(M_A : M_B)_\omega \le \sum_{i=1}^{n} \Delta_i \le n \cdot R_{\max,A}(\mathcal{N}) = n \cdot R_{\max}(\mathcal{N}). \quad (8)$$

For any achievable (r, n, ε), denote $k = 2^{nr}$

- $\mathrm{Tr}\,\Phi_k \omega_{M_A M_B} \ge 1 - \varepsilon$; $\quad \mathrm{Tr}\,\Phi_k \sigma \le \frac{1}{k}$ for any $\sigma \in \mathrm{PPT}'$ [Rains, 2001].
- perform test $\{\Phi_k, \mathbb{1} - \Phi_k\}$, $D_H^\varepsilon(\omega\|\sigma) \ge \log k$ for any $\sigma \in \mathrm{PPT}'$.
- $D_{\max}(\rho\|\sigma) \ge D_H^\varepsilon(\rho\|\sigma) + \log(1-\varepsilon)$ [Dupuis et.al, 2013].

$$R_{\max}(M_A : M_B)_\omega = \min_{\sigma \in \mathrm{PPT}'} D_{\max}(\omega\|\sigma) \ge \log(1-\varepsilon)k. \quad (9)$$

Combining Eq. (8),(9), we have $\varepsilon \ge 1 - 2^{-n(r - R_{\max}(\mathcal{N}))}$, which implies strong converse.

Known converse bounds

	Q	$Q^{\dagger}$	$Q^{\leftrightarrow}$	$Q^{\leftrightarrow,\dagger}$	Efficiently computable	General channels
$Q_{\Gamma}\,(R_{\max})$	✓	✓	✓	✓	✓	✓
R	✓	✓	?	?	? (max-min)	✓
ε-DEG	✓	?	?	?	✓	✓
E_{sq}	✓	?	✓	?	? (max-min & unbounded dim.)	✓
E_C	✓	✓	✓	✓	? (regularization)	✓
Q_E	✓	✓	✓	?	✓	✓
Q_{ss}	✓	?	?	?	? (unbounded dim.)	✓
Q_{Θ}	✓	✓	✓	✓	✓	✓

- $Q_{\Gamma}\,(R_{\max})$: SDP strong converse bound in this talk.
- R: Rains information [Tomamichel et.al, 2014]
- ε-DEG: Epsilon degradable bound [Sutter et.al, 2014]
- E_{sq}: Squashed entanglement of a channel [Takeoka et.al, 2013]
- E_C: Entanglement cost of a channel [Berta et.al, 2011]
- Q_E: Entanglement assisted quantum capacity [Bennett et.al, 2009]
- Q_{ss}: Quantum capacity with symmetric side channels [Smith et.al, 2006]
- Q_{Θ}: Partial transposition bound [Holevo,Werner, 1999; Muller-Hermes et.al, 2015]
- $\exists\,\mathcal{N},\ Q_{\Gamma}(\mathcal{N}) < \varepsilon\text{-DEG}(\mathcal{N})$; $\exists\,\mathcal{N},\ Q_{\Gamma}(\mathcal{N}) < Q_E(\mathcal{N})$.

Thanks for your attention!

See arXiv:

1709.00200 & 1709.04907

for more details