# An almost complete picture of quantum hypothesis testing with composite correlated hypotheses

arXiv:2508.12901 & 2508.12889

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Shaanxi Normal University, November 2025 (online)

# What is “hypothesis testing”?

Null hypothesis: $H_0$

Alternative hypothesis: $H_1$

**Black box**

Samples

observed data

Decide

$H_0$

$H_1$

A rigorous framework for deciding between competing explanations for observed data.
Cornerstone of statistics, information theory, signal processing, machine learning, and experimental physics…

$H_0$: Healthy

$H_1$: Sick

Samples

Decide

Negative

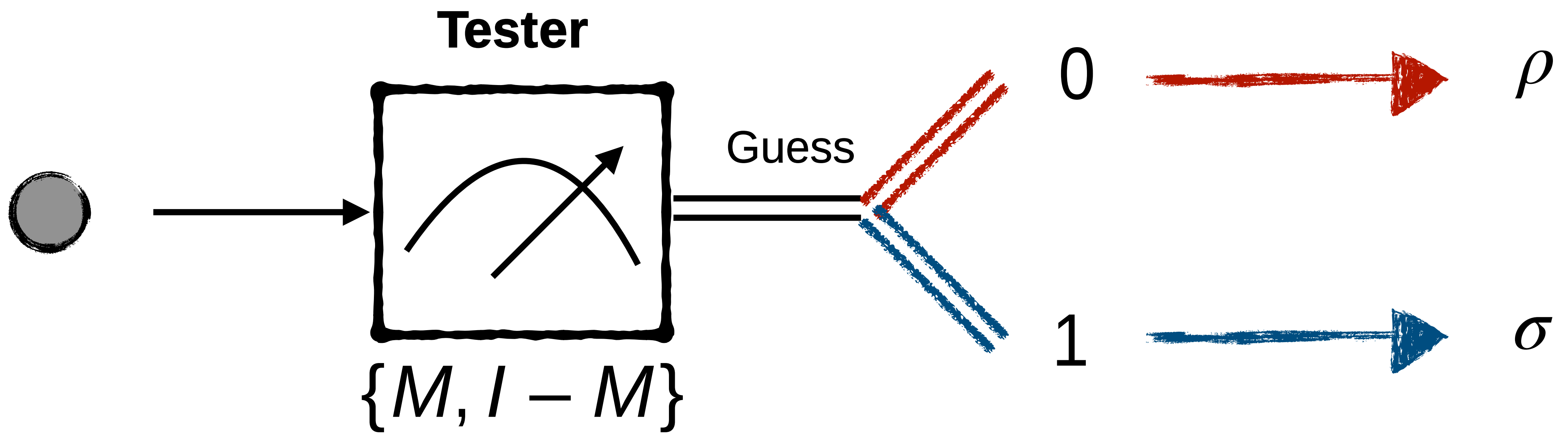
Positive

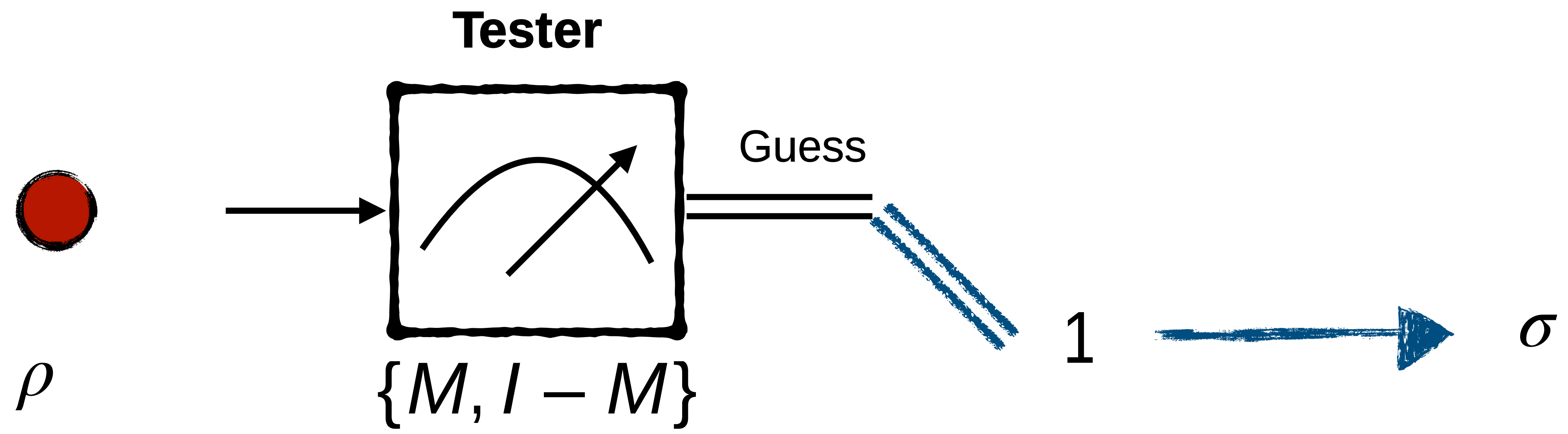
Primary objective: **identify** the best **model**, while **minimizing the probabilities of errors** (i.e., type-I and type-II errors).

# What is “quantum hypothesis testing”?

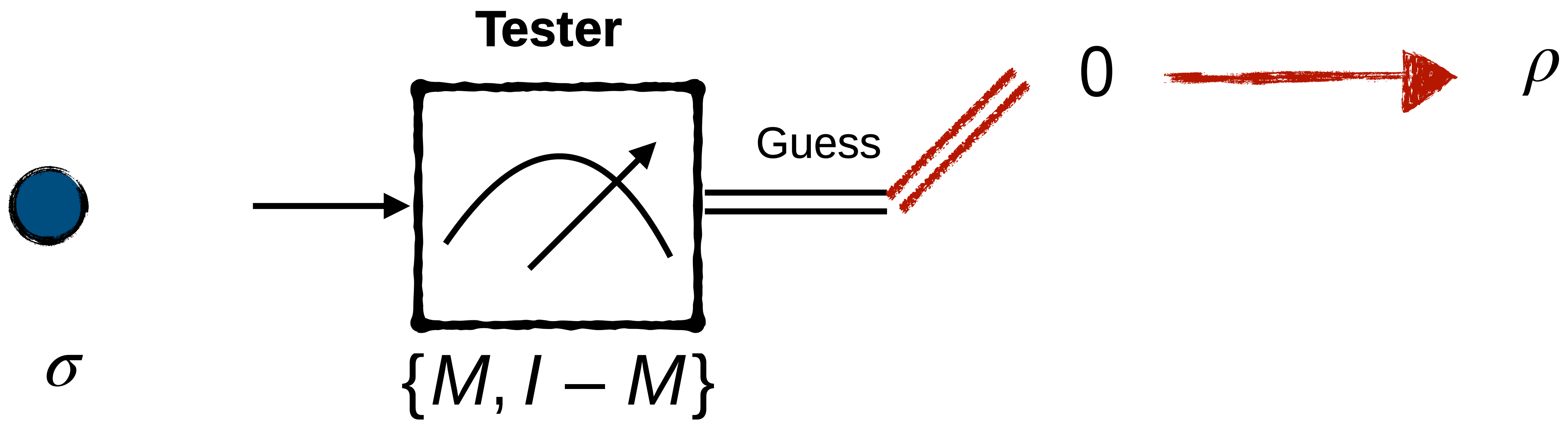
Null hypothesis: $\rho$

Alternative hypothesis: $\sigma$



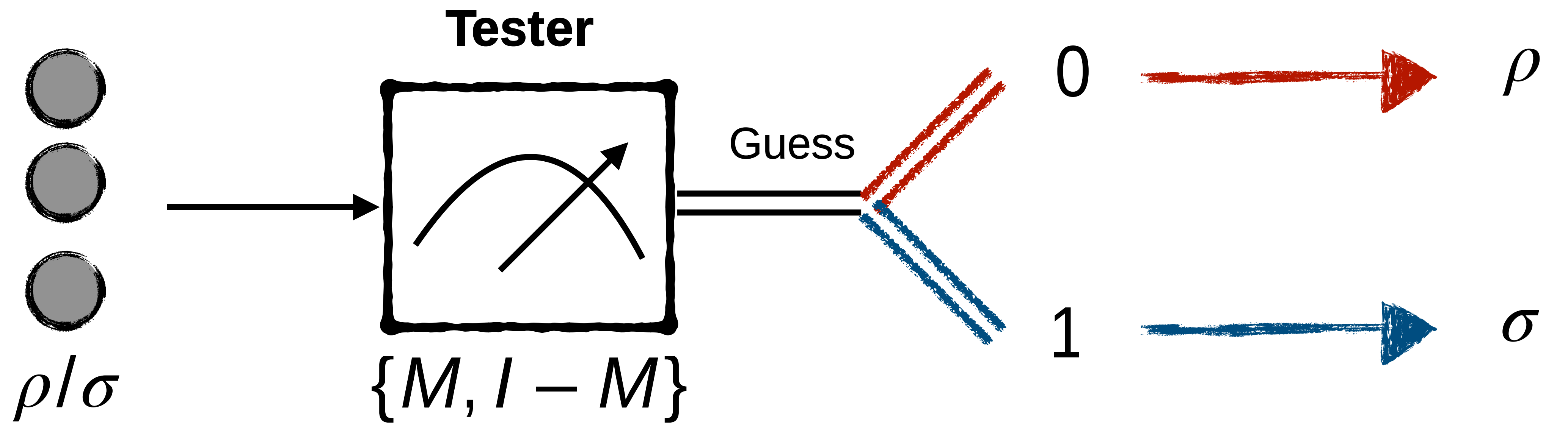

**Type-I error** $\alpha(\rho, M) := \mathrm{Tr}\,[\rho(I - M)]$

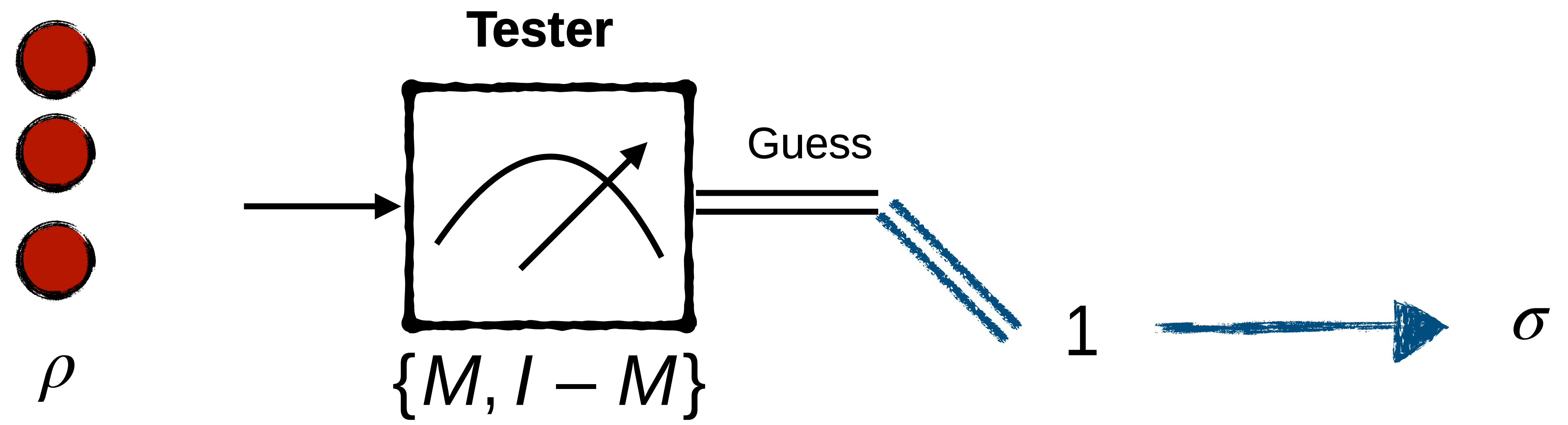



**Type-II error** $\beta(\sigma, M) := \mathrm{Tr}\,[\sigma M]$

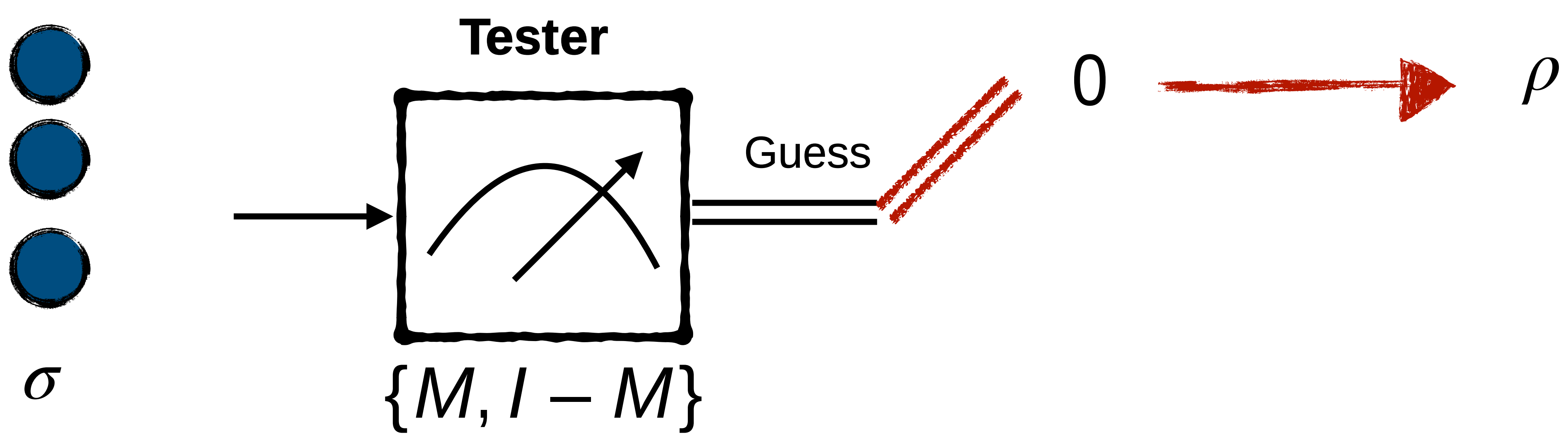
Primary objective: **identify** the best **model**, while **minimizing the probabilities of errors** (i.e., type-I and type-II errors).

# What is "quantum hypothesis testing"?




**Type-I error** $\alpha(\rho^{\otimes n}, M_n) := \mathrm{Tr}\,[\rho^{\otimes n}(I - M_n)]$

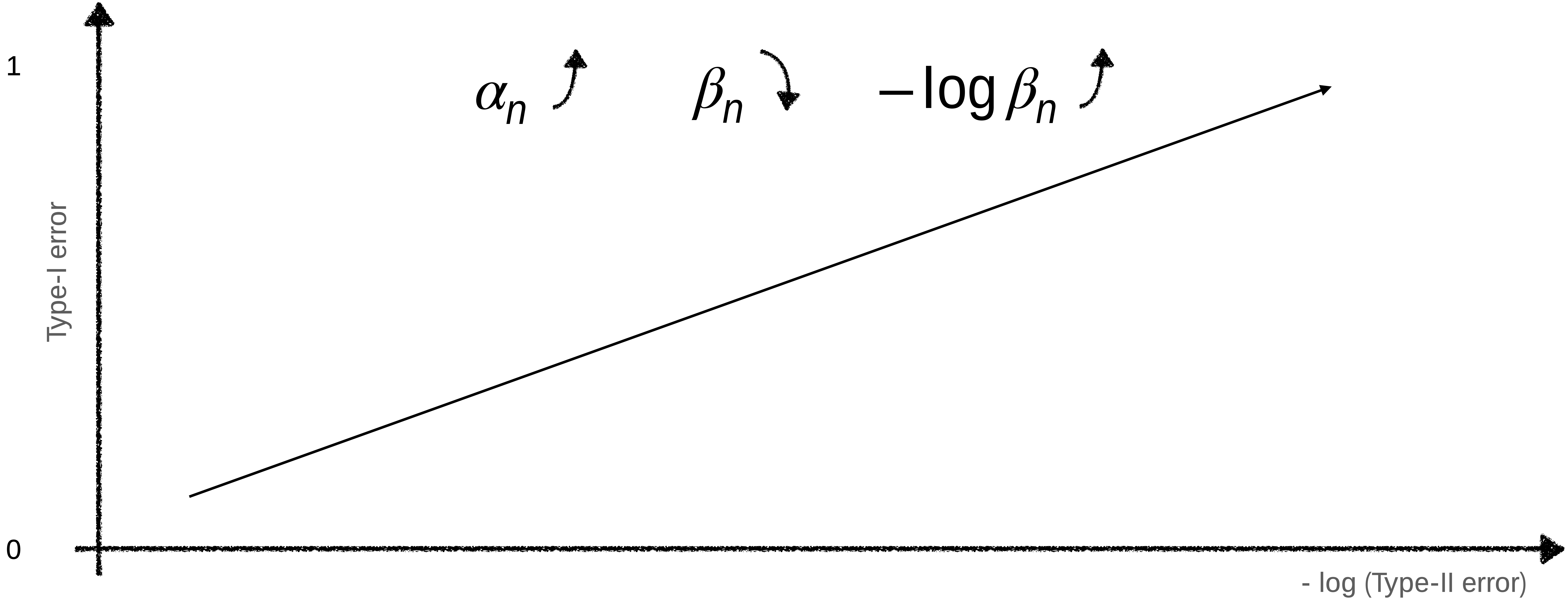



**Type-II error** $\beta(\sigma^{\otimes n}, M_n) := \mathrm{Tr}\,[\sigma^{\otimes n} M_n]$

Primary objective: **identify** the best **model**, while **minimizing the probabilities of errors** (i.e., type-I and type-II errors).

# Different operational regimes for error tradeoff

**Identify** the best **model**, while **minimizing the probabilities of errors**



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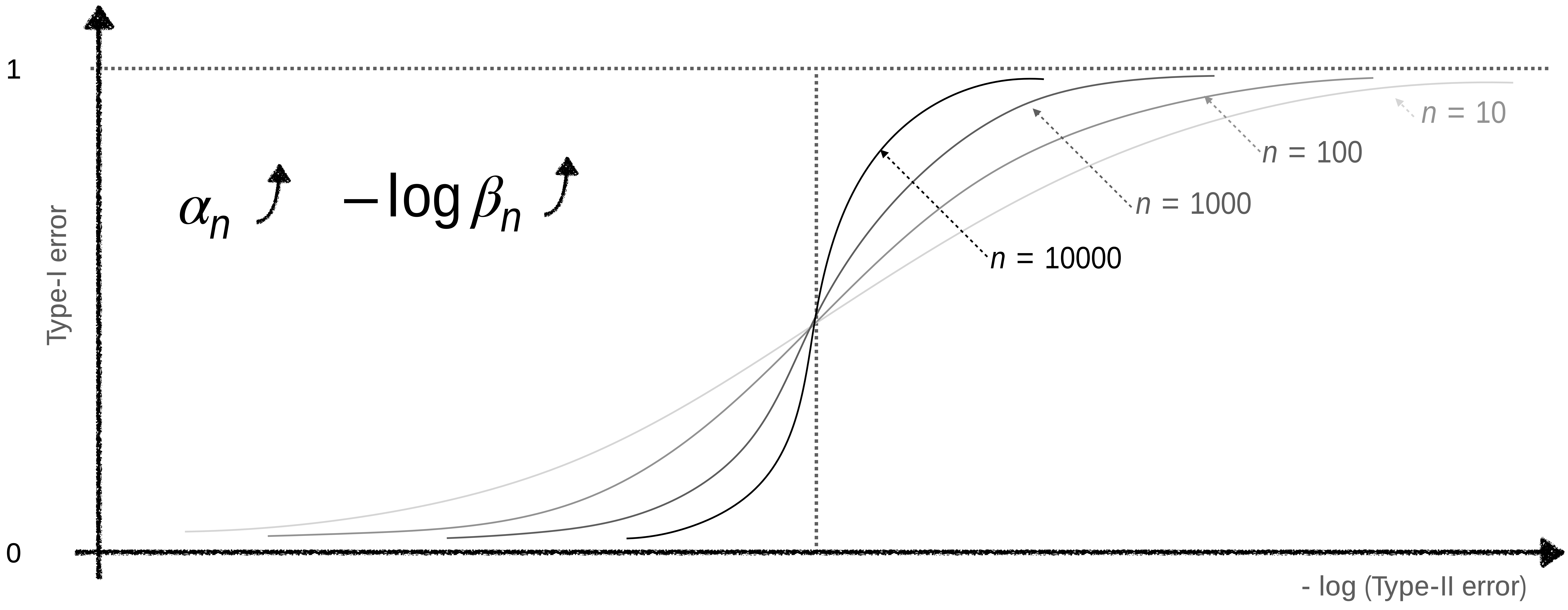

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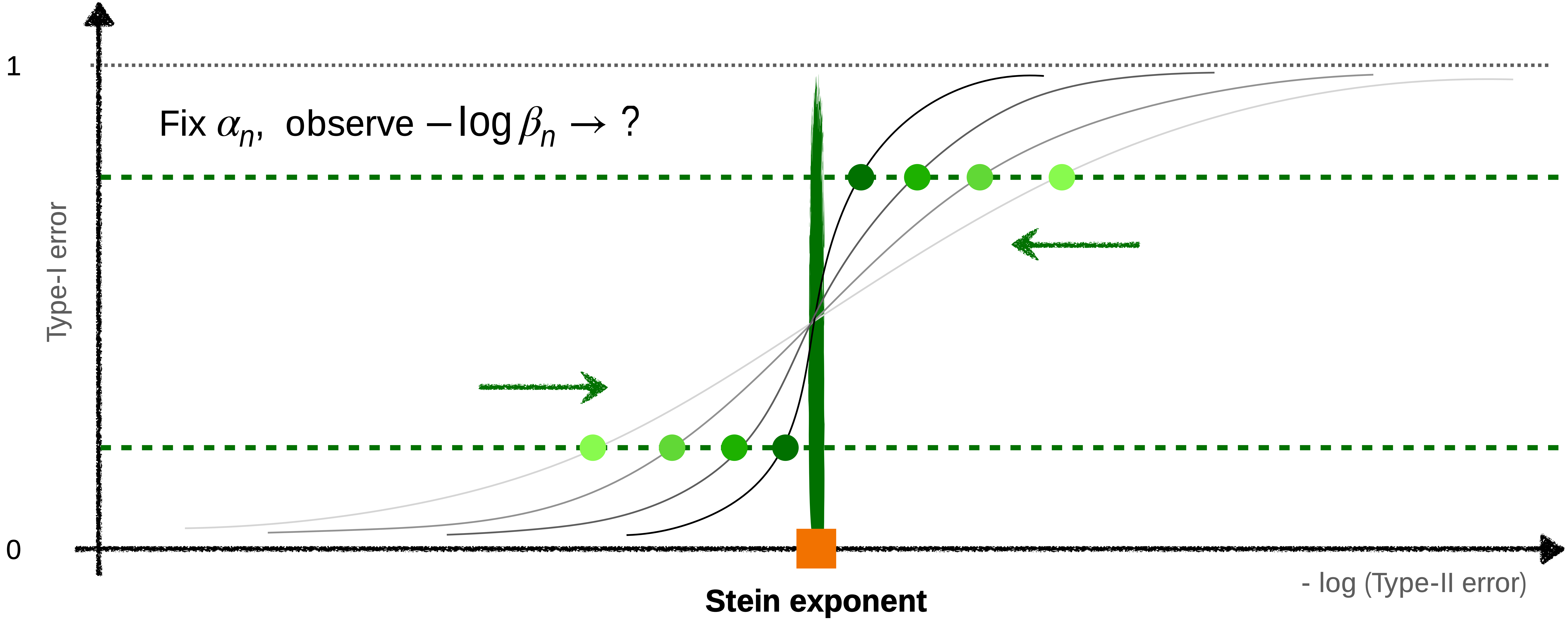


# Different operational regimes for error tradeoff

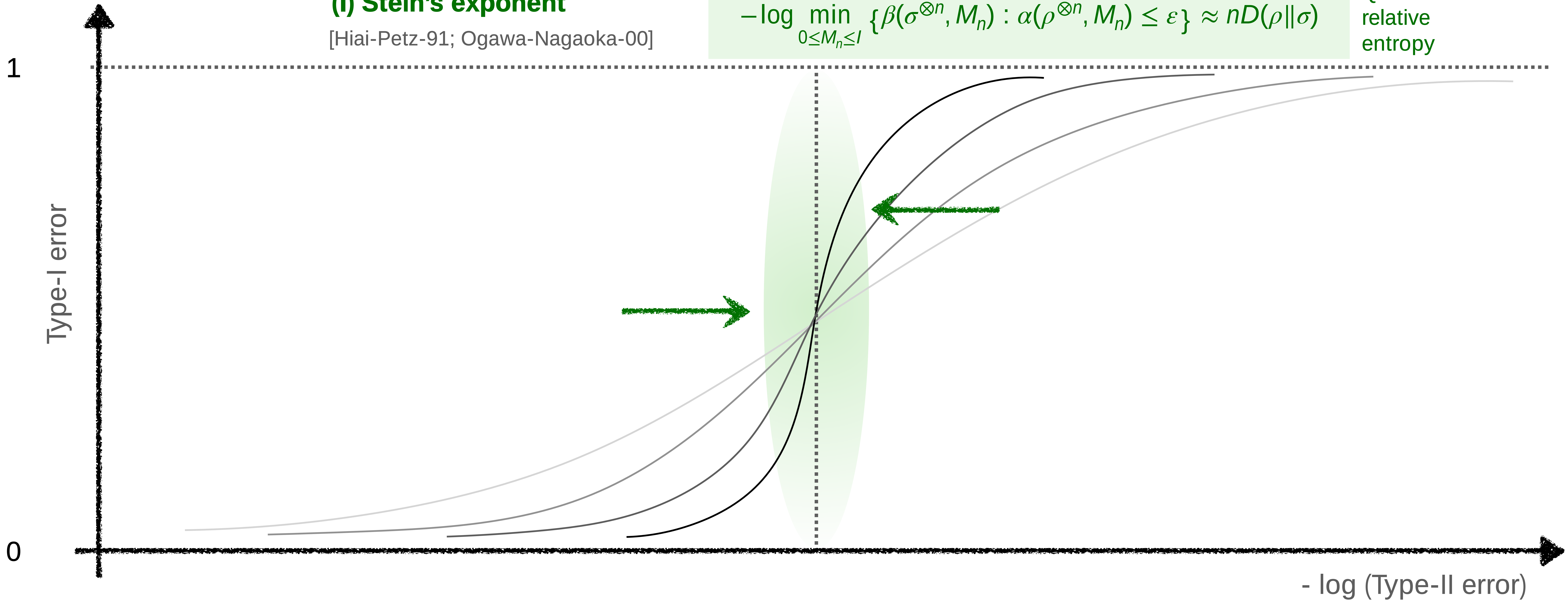
**Identify** the best **model**, while **minimizing the probabilities of errors**

**(I) Stein's exponent**

[Hiai-Petz-91; Ogawa-Nagaoka-00]

$$-\log \min_{0\le M_n \le I} \{\beta(\sigma^{\otimes n}, M_n) : \alpha(\rho^{\otimes n}, M_n) \le \varepsilon\} \approx nD(\rho\|\sigma)$$

Quantum relative entropy



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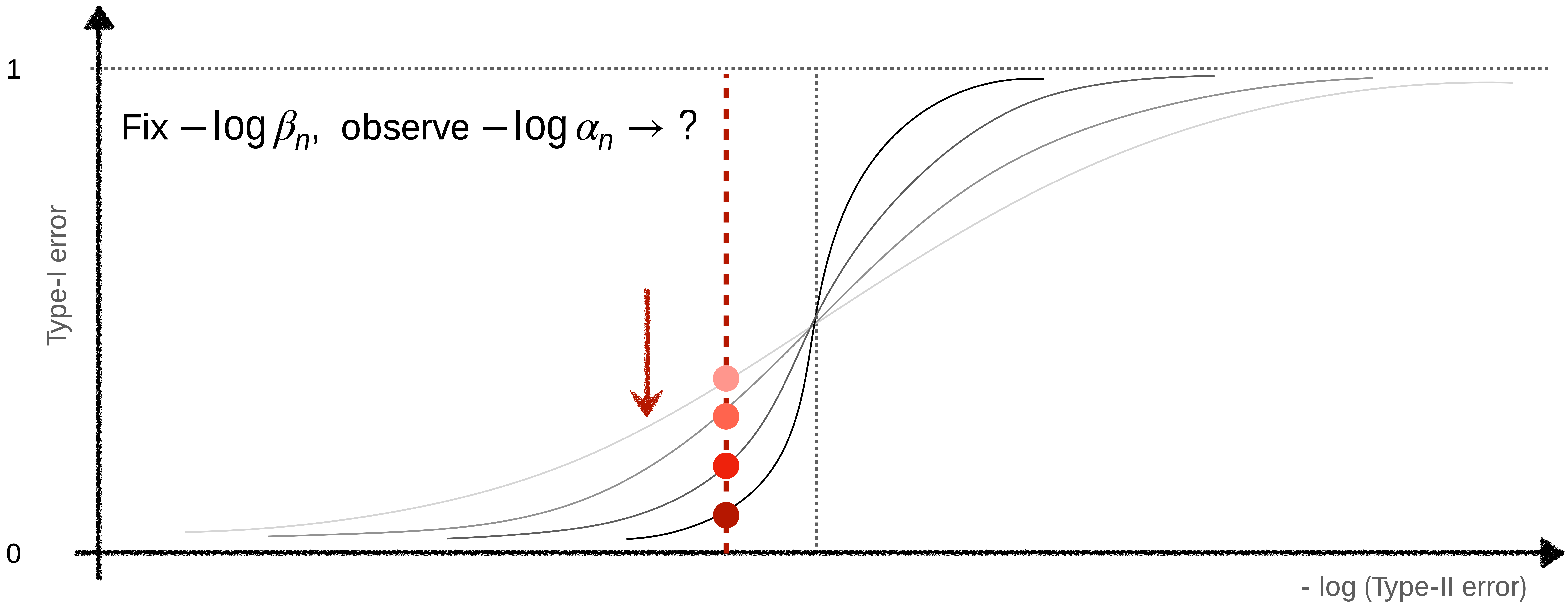

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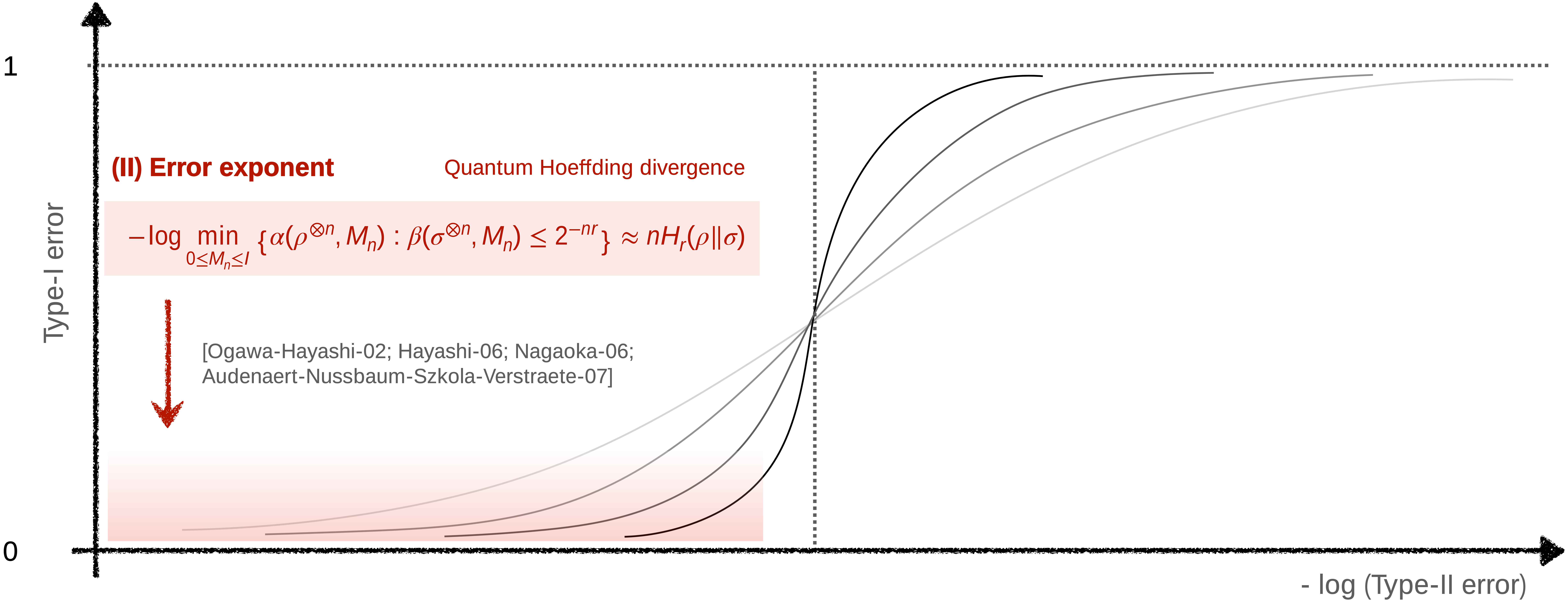

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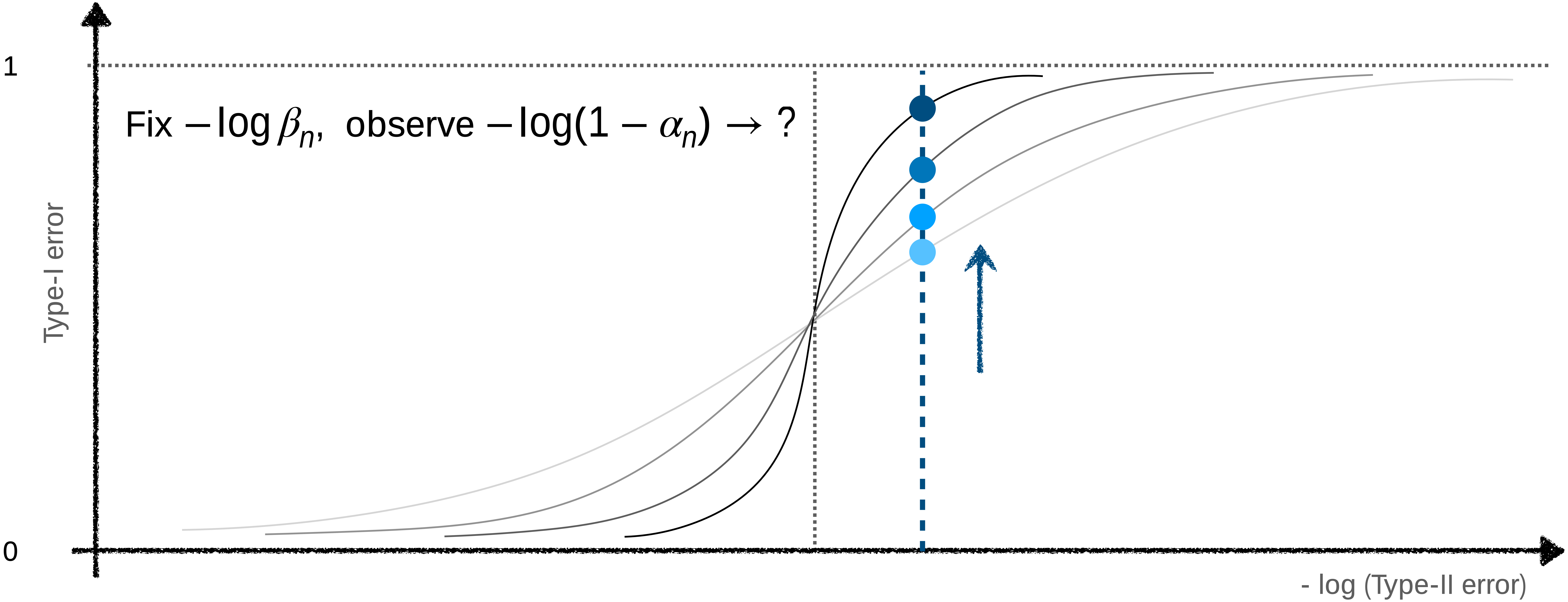

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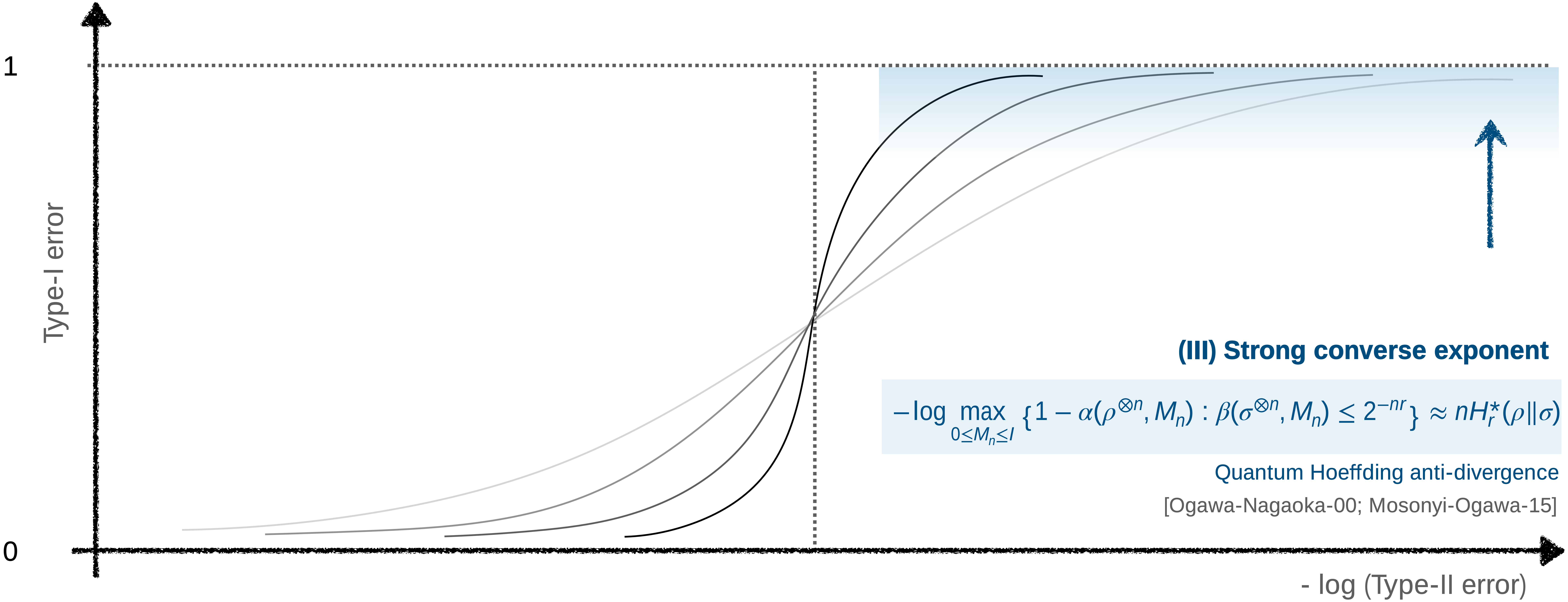



**(III) Strong converse exponent**

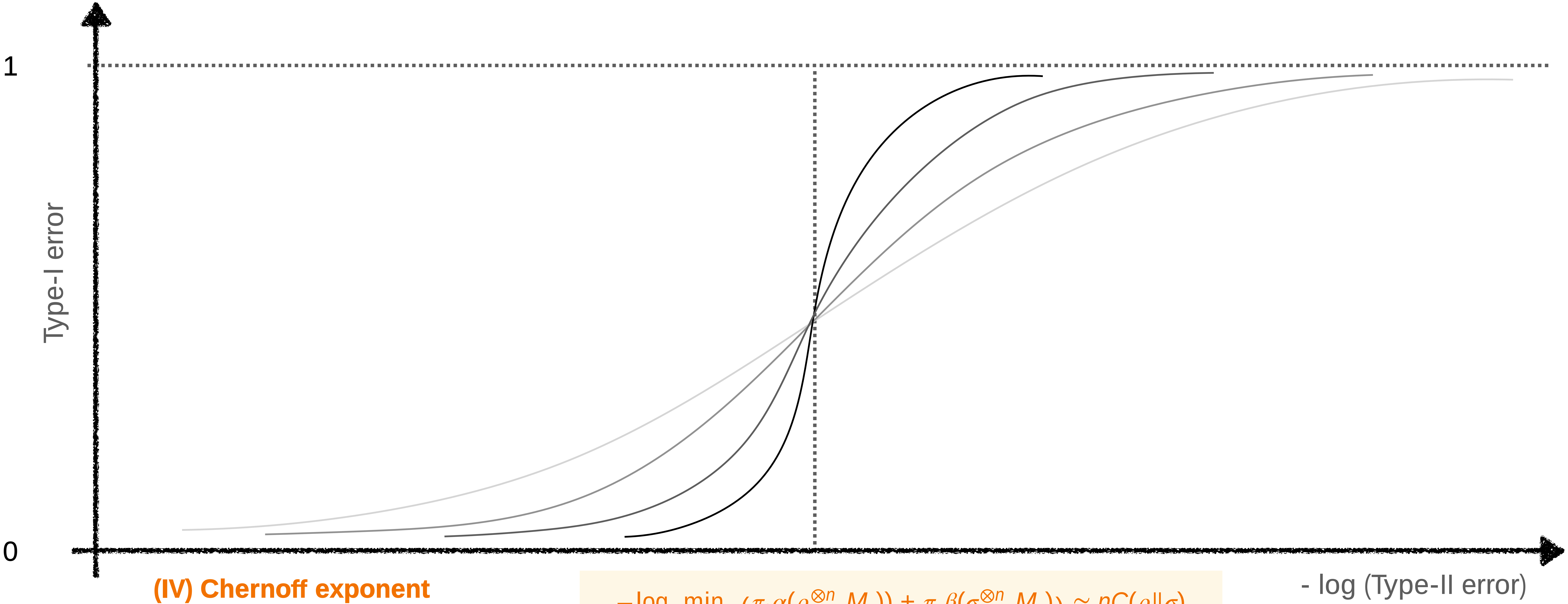
$$-\log \max_{0\leq M_n\leq I} \{1-\alpha(\rho^{\otimes n}, M_n) : \beta(\sigma^{\otimes n}, M_n) \leq 2^{-nr}\} \approx nH_r^*(\rho\|\sigma)$$

Quantum Hoeffding anti-divergence

[Ogawa-Nagaoka-00; Mosonyi-Ogawa-15]

# Different operational regimes for error tradeoff

**Identify** the best **model**, while **minimizing the probabilities of errors**




**(IV) Chernoff exponent**

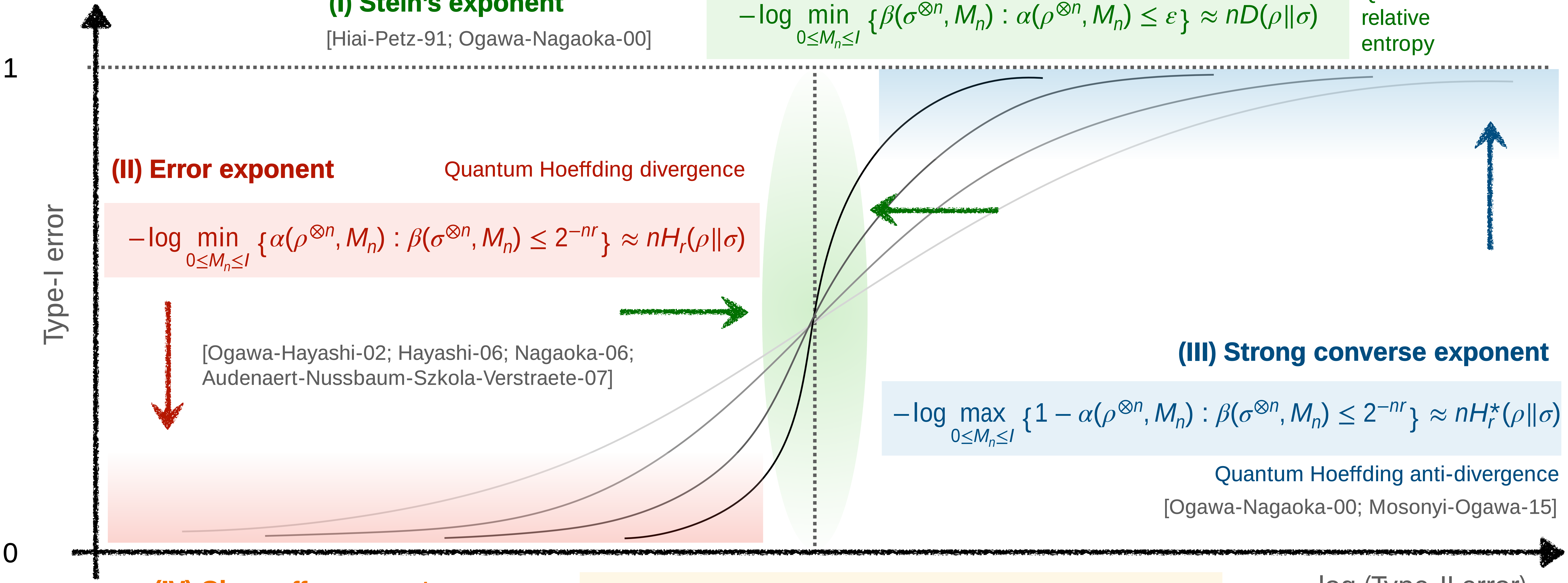
[Audenaert et al.-07; Nussbaum-Szkoła-09]

$$-\log \min_{0 \leq M_n \leq I} (\pi_0 \alpha(\rho^{\otimes n}, M_n)) + \pi_1 \beta(\sigma^{\otimes n}, M_n)) \approx nC(\rho \| \sigma)$$

Quantum Chernoff divergence

# Different operational regimes for error tradeoff

**Identify** the best **model**, while **minimizing the probabilities of errors**




**(I) Stein's exponent**

[Hiai-Petz-91; Ogawa-Nagaoka-00]

$$-\log \min_{0\le M_n \le I} \{\beta(\sigma^{\otimes n}, M_n) : \alpha(\rho^{\otimes n}, M_n) \le \varepsilon\} \approx nD(\rho\|\sigma)$$

Quantum relative entropy

**(II) Error exponent**

Quantum Hoeffding divergence

$$-\log \min_{0\le M_n \le I} \{\alpha(\rho^{\otimes n}, M_n) : \beta(\sigma^{\otimes n}, M_n) \le 2^{-nr}\} \approx nH_r(\rho\|\sigma)$$

[Ogawa-Hayashi-02; Hayashi-06; Nagaoka-06; Audenaert-Nussbaum-Szkola-Verstraete-07]

**(III) Strong converse exponent**

$$-\log \max_{0\le M_n \le I} \{1-\alpha(\rho^{\otimes n}, M_n) : \beta(\sigma^{\otimes n}, M_n) \le 2^{-nr}\} \approx nH_r^*(\rho\|\sigma)$$

Quantum Hoeffding anti-divergence

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**(IV) Chernoff exponent**

[Audenaert et al.-07; Nussbaum-Szkoła-09]

$$-\log \min_{0\le M_n \le I} \left(\pi_0\alpha(\rho^{\otimes n}, M_n) + \pi_1\beta(\sigma^{\otimes n}, M_n)\right) \approx nC(\rho\|\sigma)$$

Quantum Chernoff divergence

# Different operational regimes for error tradeoff

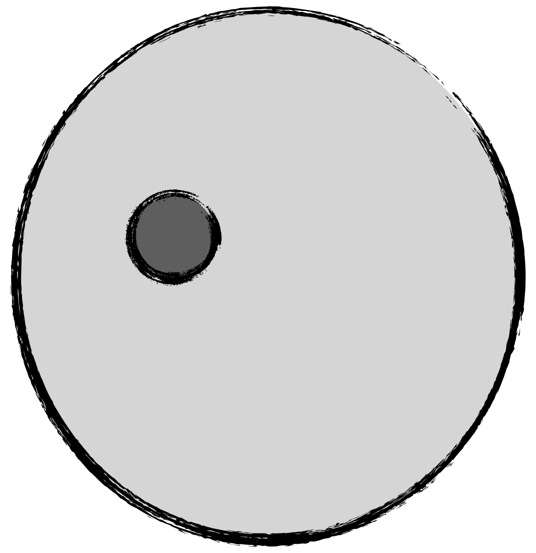
**Identify** the best **model**, while **minimizing the probabilities of errors**

| | | |
|---|---|---|
| **(I) Stein's exponent** | Quantum relative entropy | [Hiai-Petz-91; Ogawa-Nagaoka-00] |
| **(II) Error exponent** | Quantum Hoeffding divergence | [Ogawa-Hayashi-02; Hayashi-06; Nagaoka-06; Audenaert-Nussbaum-Szkola-Verstraete-07] |
| **(III) Strong converse exponent** | Quantum Hoeffding anti-divergence | [Ogawa-Nagaoka-00; Mosonyi-Ogawa-15] |
| **(IV) Chernoff exponent** | Quantum Chernoff divergence | [Audenaert et al.-07; Nussbaum-Szkoła-09] |

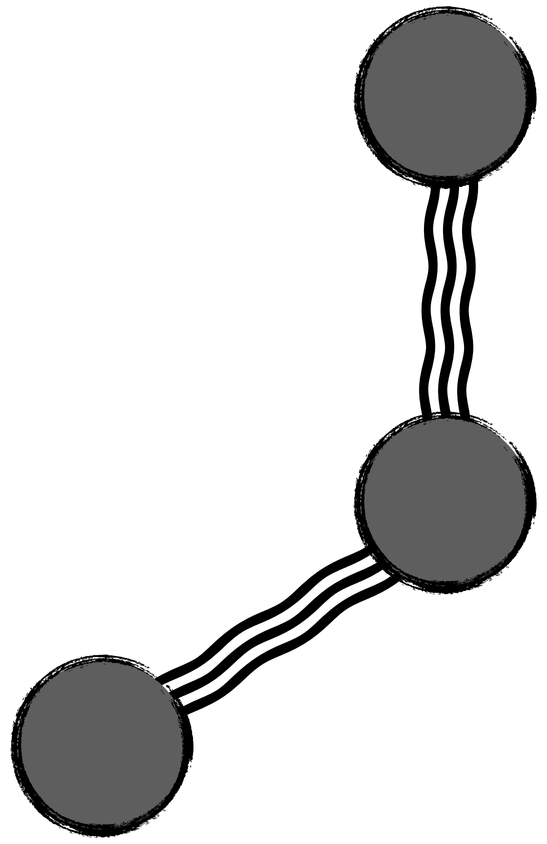
**A complete understanding of quantum hypothesis testing for i.i.d. sources**

# What about “beyond i.i.d.” sources?

While much of the existing literature has focused on i.i.d. sources, practical scenarios often involve



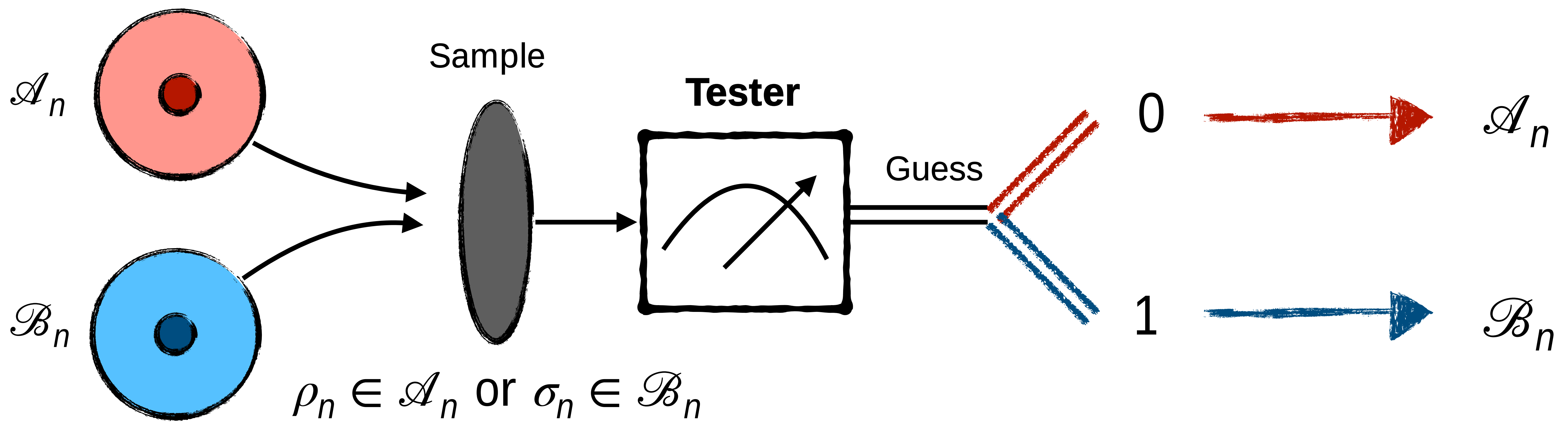
- **Composite hypotheses**: states are not fully specified
  - [Brandao-Plenio-10]: $\rho^{\otimes n}$ v.s. SEP;
  - [Berta-Brandao-Hirche-21]: $\left\{ \int \rho^{\otimes n} d\nu(\rho) \,|\, \rho \in S \right\}$ v.s. $\left\{ \int \sigma^{\otimes n} d\mu(\sigma) \,|\, \sigma \in T \right\}$;
  - [Mosonyi-Szilagyi-Weiner-22]: $\{\rho^{\otimes n} \,|\, \rho \in S\}$ v.s. $\{\sigma^{\otimes n} \,|\, \sigma \in T\}$



- **Correlated hypotheses**: states are correlated
  - [Hiai-Mosonyi-Ogawa-07,08]: $\rho_n$ v.s. $\sigma_n$, correlated states on a spin chain
  - [Mosonyi-Ogawa-15]: $\rho_n$ v.s. $\sigma_n$, correlated states on a spin chain

# What is “Composite Correlated” sources?

A tester draws samples from *two sets of correlated quantum states*,
and performs measurements to determine which set the sample belongs to.




**(I) Stein’s exponent ?**

**(II) Error exponent?**

**(III) Strong converse exponent?**

**(IV) Chernoff exponent?**

**Type-I error** $\alpha(\mathscr{A}_n, M_n) := \sup_{\rho_n \in \mathscr{A}_n} \operatorname{Tr}[\rho_n(I - M_n)]$

Worst-case

**Type-II error** $\beta(\mathscr{B}_n, M_n) := \sup_{\sigma_n \in \mathscr{B}_n} \operatorname{Tr}[\sigma_n M_n]$

# What is "Composite Correlated" sources?

A tester draws samples from *two sets of correlated quantum states*,
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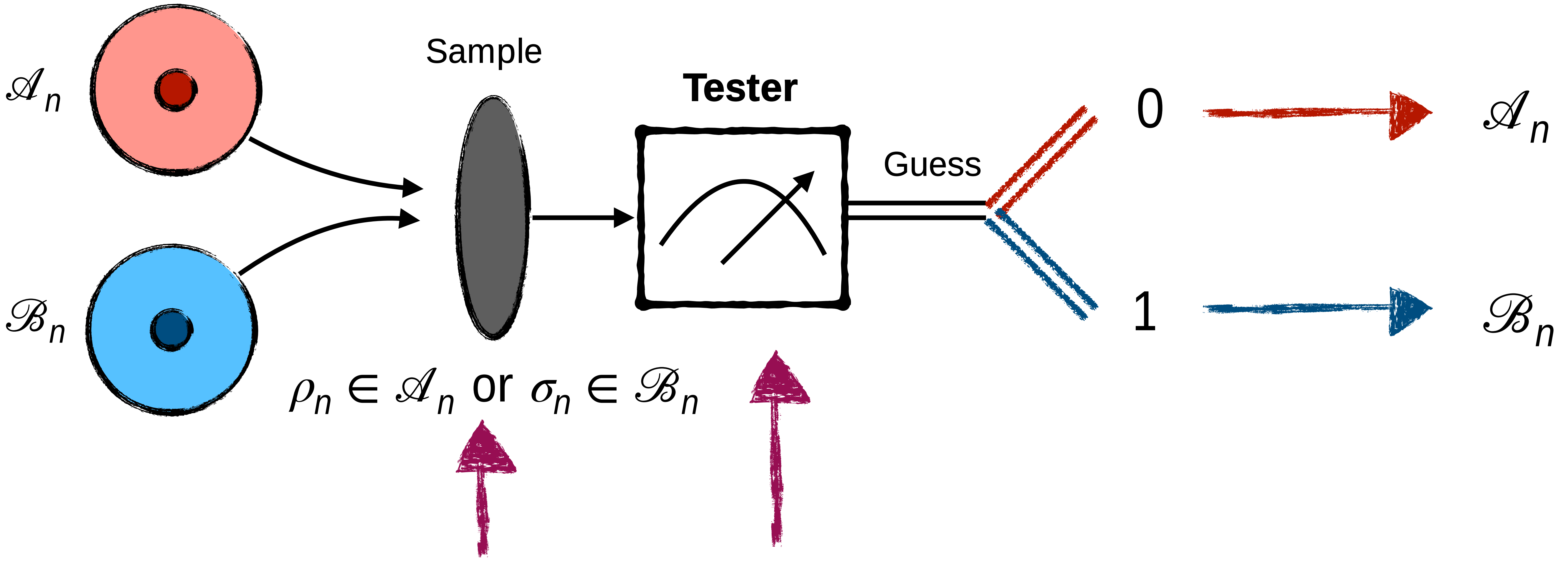



non-i.i.d. structure

state-agnostic test

Minimax opt.: Max over states v.s. Min over test

**Challenges**

**(I) Stein's exponent ?**

**(II) Error exponent?**

**(III) Strong converse exponent?**

**(IV) Chernoff exponent?**

# What is “Composite Correlated” sources?

A tester draws samples from *two sets of correlated quantum states*,
and performs measurements to determine which set the sample belongs to.

## (I) Stein’s exponent   (recent progress)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| [Hayashi-Yamasaki-24] | $\rho^{\otimes n}$ | $\mathscr{B}_n$ | A.1 | | A.3 | | A.5 | |
| [Lami-24] | $\rho^{\otimes n}$ | $\mathscr{B}_n$ | A.1 | A.2 | A.3 | | A.5 | A.6 |
| [KF-Fawzi-Fawzi-24 ] | $\mathscr{A}_n$ | $\mathscr{B}_n$ | A.1 | A.2 | A.3 | A.4 | | |

(A.1) Each $\mathscr{B}_n$ is convex and compact;

(A.2) Each $\mathscr{B}_n$ is permutation-invariant;

(A.3) $\mathscr{B}_m \otimes \mathscr{B}_k \subseteq \mathscr{B}_{m+k}$, for all $m, k \in \mathbb{N}$;

(A.5) $\mathscr{B}_1$ contains a full-rank state

(A.6) Each $\mathscr{B}_n$ is closed under partial traces

(A.4) $(\mathscr{B}_m)^{\circ}_{+} \otimes (\mathscr{B}_k)^{\circ}_{+} \subseteq (\mathscr{B}_{m+k})^{\circ}_{+}$, for all $m, k \in \mathbb{N}$;

Extending the Stein exponent to more general setting and developing **new applications** across

- quantum cryptography [KF-Fawzi-Fawzi-24]
- quantum communication [Cao-Yao-Berta-25]
- quantum resource theory [KF-Fawzi-Fawzi-24]

# What is “Composite Correlated” sources?

A tester draws samples from *two sets of correlated quantum states*,
and performs measurements to determine which set the sample belongs to.

**(I) Stein’s exponent (recent progress)**

**This talk**

**(II) Error exponent (2508.12901)**

**(III) Strong converse exponent (2508.12901)**

**(IV) Chernoff exponent (2508.12889 )**

an almost complete picture

(A.1) Each $\mathscr{B}_n$ is convex and compact;

~~(A.2) Each $\mathscr{B}_n$ is permutation-invariant;~~

(A.3) $\mathscr{B}_m \otimes \mathscr{B}_k \subseteq \mathscr{B}_{m+k}$, for all $m, k \in \mathbb{N}$;

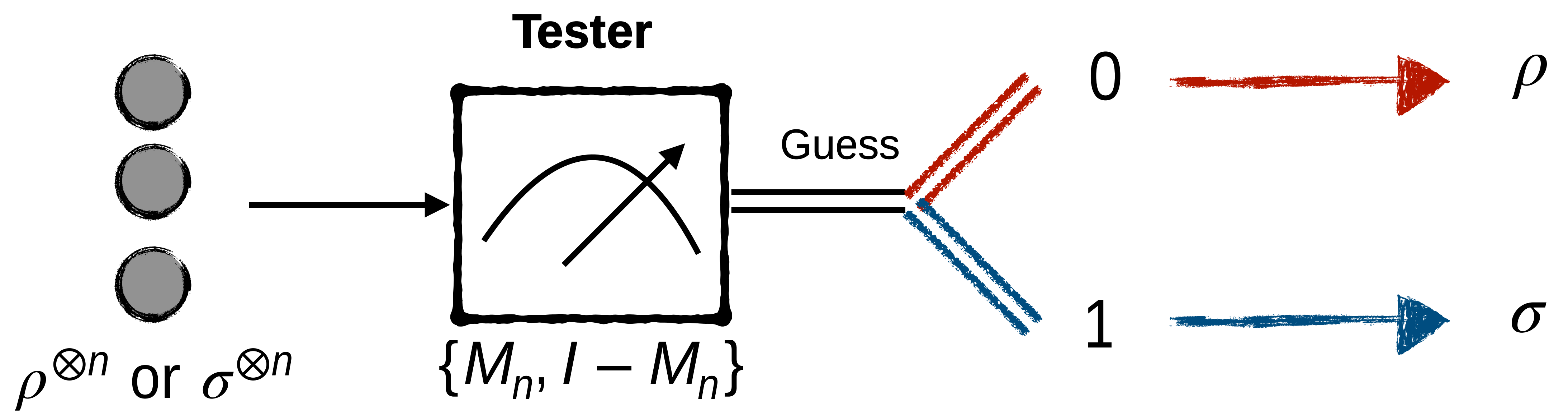
~~(A.5) $\mathscr{B}_1$ contains a full-rank state~~

~~(A.6) Each $\mathscr{B}_n$ is closed under partial traces~~

~~(A.4) $(\mathscr{B}_m)^\circ_+ \otimes (\mathscr{B}_k)^\circ_+ \subseteq (\mathscr{B}_{m+k})^\circ_+$, for all $m, k \in \mathbb{N}$;~~

Only two assumptions (A.1) and (A.3) are required.

# (II) Error exponent (2508.12901)



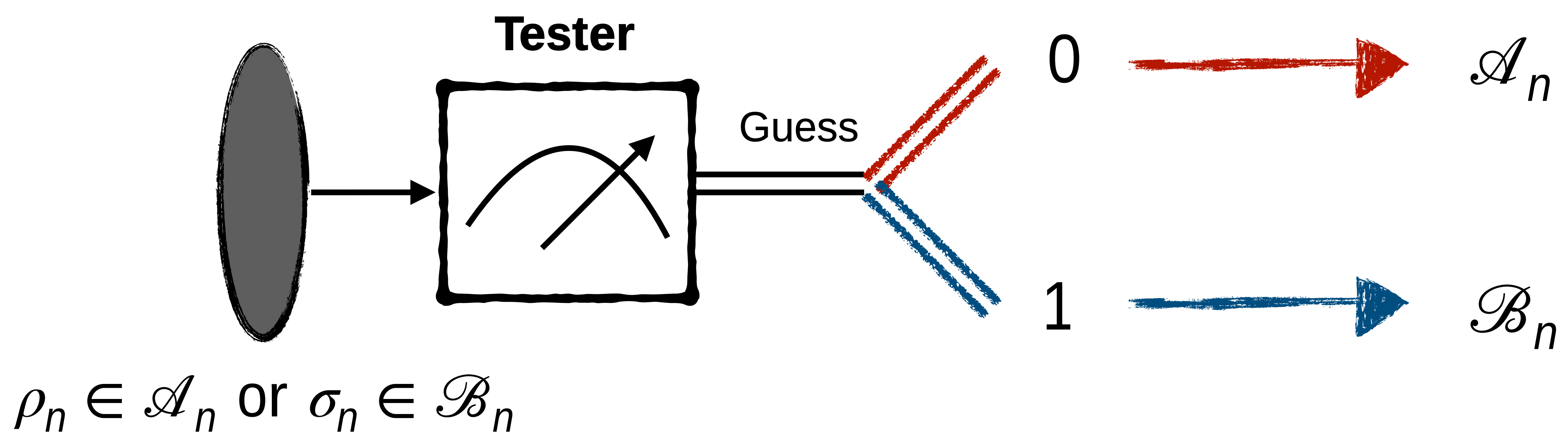

$$\alpha_{n,r}(\rho_n \| \sigma_n) := \min_{0 \le M_n \le I} \{\alpha(\rho_n, M_n) : \beta(\sigma_n, M_n) \le 2^{-nr}\}$$

$$\lim_{n\to\infty} -\frac{1}{n} \log \alpha_{n,r}(\rho^{\otimes n}, \sigma^{\otimes n}) = H_r(\rho\|\sigma)$$

**Quantum Hoeffding divergence**

$$H_r(\rho\|\sigma) := \sup_{\alpha\in(0,1)} \frac{\alpha - 1}{\alpha} \left(r - D_{\mathrm{P},\alpha}(\rho\|\sigma)\right)$$

**Petz** $\quad D_{\mathrm{P},\alpha}(\rho\|\sigma) := \frac{1}{\alpha - 1} \log \mathrm{Tr}\,[\rho^\alpha \sigma^{1-\alpha}]$




$$\alpha_{n,r}(\mathscr{A}_n \| \mathscr{B}_n) := \min_{0 \le M_n \le I} \{\alpha(\mathscr{A}_n, M_n) : \beta(\mathscr{B}_n, M_n) \le 2^{-nr}\}$$

$$\lim_{n\to\infty} -\frac{1}{n} \log \alpha_{n,r}(\mathscr{A}_n, \mathscr{B}_n) \overset{?}{=} H_r(\mathscr{A}\|\mathscr{B})$$

**How to define $H_r(\mathscr{A}\|\mathscr{B})$ ?**

# (II) Error exponent (2508.12901)

## How to define $H_r(\mathscr{A}\|\mathscr{B})$ ?

$D_{\mathrm{P},\alpha}(\mathscr{A}_n\|\mathscr{B}_n) := \inf_{\rho_n\in\mathscr{A}_n,\, \sigma_n\in\mathscr{B}_n} D_{\mathrm{P},\alpha}(\rho_n\|\sigma_n)$

$$H_{n,r}(\rho_n\|\sigma_n) := \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\left(nr - D_{\mathrm{P},\alpha}(\rho_n\|\sigma_n)\right)$$

Extension 1

$$H_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \inf_{\rho_n\in\mathscr{A}_n,\, \sigma_n\in\mathscr{B}_n} H_{n,r}(\rho_n\|\sigma_n)$$

Extension 2

$$\mathfrak{H}_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\left(nr - D_{\mathrm{P},\alpha}(\mathscr{A}_n\|\mathscr{B}_n)\right)$$

**Are they equivalent?**

$$\inf_{\rho_n\in\mathscr{A}_n,\, \sigma_n\in\mathscr{B}_n} \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\left(nr - D_{\mathrm{P},\alpha}(\rho_n\|\sigma_n)\right) \underset{\text{minimax theorem}}{=\!=} \sup_{\alpha\in(0,1)} \inf_{\rho_n\in\mathscr{A}_n,\, \sigma_n\in\mathscr{B}_n} \frac{\alpha-1}{\alpha}\left(nr - D_{\mathrm{P},\alpha}(\rho_n\|\sigma_n)\right)$$

Convex in $(\rho_n, \sigma_n)$

Replace $u = (\alpha-1)/\alpha$, then concave in $u$

# (II) Error exponent (2508.12901)

## How to define $H_r(\mathscr{A}\|\mathscr{B})$ ?

$$D_{\mathrm{P},\alpha}(\mathscr{A}_n\|\mathscr{B}_n) := \inf_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} D_{\mathrm{P},\alpha}(\rho_n\|\sigma_n)$$

$$H_{n,r}(\rho_n\|\sigma_n) := \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\left(nr - D_{\mathrm{P},\alpha}(\rho_n\|\sigma_n)\right)$$

Extension 1

Extension 2

$$H_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \inf_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} H_{n,r}(\rho_n\|\sigma_n)$$

$$\mathfrak{H}_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\left(nr - D_{\mathrm{P},\alpha}(\mathscr{A}_n\|\mathscr{B}_n)\right)$$

# (II) Error exponent (2508.12901)

## How to define $H_r(\mathscr{A}\|\mathscr{B})$ ?

$$H_{n,r}(\rho_n\|\sigma_n) := \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\left(nr - D_{\mathrm{P},\alpha}(\rho_n\|\sigma_n)\right)$$

Extension 1

$$H_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \inf_{\rho_n\in\mathscr{A}_n,\,\sigma_n\in\mathscr{B}_n} H_{n,r}(\rho_n\|\sigma_n)$$

$$H_r^\infty(\mathscr{A}\|\mathscr{B}) := \lim_{n\to\infty}\frac{1}{n}H_{n,r}(\mathscr{A}_n\|\mathscr{B}_n)$$

Extension 2

$$\mathfrak{H}_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\left(nr - D_{\mathrm{P},\alpha}(\mathscr{A}_n\|\mathscr{B}_n)\right)$$

$$\mathfrak{H}_r^\infty(\mathscr{A}\|\mathscr{B}) := \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\left(r - D_{\mathrm{P},\alpha}^\infty(\mathscr{A}\|\mathscr{B})\right)$$

**Are they equivalent?**

$$\inf_{n\geq 1}\sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\left(r - \frac{1}{n}D_{\mathrm{P},\alpha}(\mathscr{A}_n\|\mathscr{B}_n)\right) \geq \sup_{\alpha\in(0,1)}\inf_{n\geq 1} \frac{\alpha-1}{\alpha}\left(r - \frac{1}{n}D_{\mathrm{P},\alpha}(\mathscr{A}_n\|\mathscr{B}_n)\right)$$

minimax inequality

Challenge for equality: both sets are non-compact

# (II) Error exponent (2508.12901)

## How to define $H_r(\mathscr{A}\|\mathscr{B})$ ?

$$H_{n,r}(\rho_n\|\sigma_n) := \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\left(nr - D_{\mathrm{P},\alpha}(\rho_n\|\sigma_n)\right)$$

Extension 1

$$H_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \inf_{\rho_n\in\mathscr{A}_n,\,\sigma_n\in\mathscr{B}_n} H_{n,r}(\rho_n\|\sigma_n)$$

$$H_r^{\infty}(\mathscr{A}\|\mathscr{B}) := \lim_{n\to\infty}\frac{1}{n}H_{n,r}(\mathscr{A}_n\|\mathscr{B}_n)$$

Extension 2

$$\mathfrak{H}_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\left(nr - D_{\mathrm{P},\alpha}(\mathscr{A}_n\|\mathscr{B}_n)\right)$$

$$\mathfrak{H}_r^{\infty}(\mathscr{A}\|\mathscr{B}) := \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\left(r - D^{\infty}_{\mathrm{P},\alpha}(\mathscr{A}\|\mathscr{B})\right)$$

**Which one characterizes the error exponent?**

Error exponent $\qquad \lim_{n\to\infty} -\frac{1}{n}\log \alpha_{n,r}(\mathscr{A}_n\|\mathscr{B}_n)$

# (II) Error exponent (2508.12901)

Error exponent $$\lim_{n\to\infty} -\frac{1}{n}\log \alpha_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) = H_r^{\infty}(\mathscr{A}\|\mathscr{B})$$

**Proof of the lower bound:** $\mathrm{Tr}[V^{\alpha}W^{1-\alpha}] \geq \mathrm{Tr}W\{W \leq V\} + \mathrm{Tr}V\{W > V\}$ [Audenaert et al.-07 ]

**Proof of the upper bound:**

$$\lim_{n\to\infty} -\frac{1}{n}\log \alpha_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) \leq \lim_{n\to\infty} -\frac{1}{mn}\log \alpha_{mn,r}(\mathscr{A}_{mn},\mathscr{B}_{mn})$$ Property of lower limit

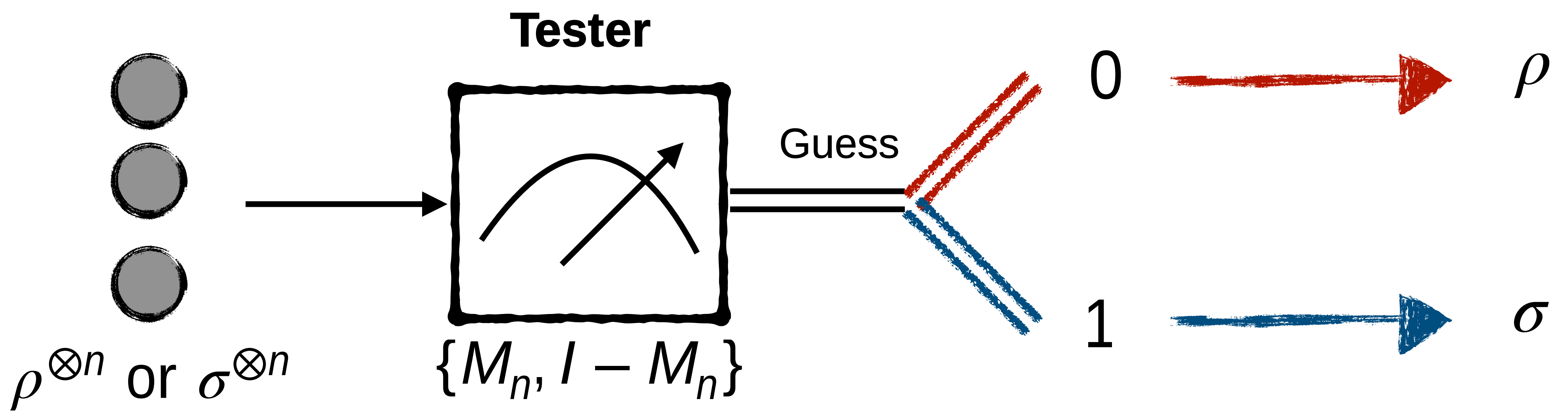
Minimax theorem $$= \lim_{n\to\infty} -\frac{1}{mn}\log \sup_{\substack{\rho_{mn}\in\mathscr{A}_{mn}\\ \sigma_{mn}\in\mathscr{B}_{mn}}} \alpha_{mn,r}(\rho_{mn},\sigma_{mn})$$

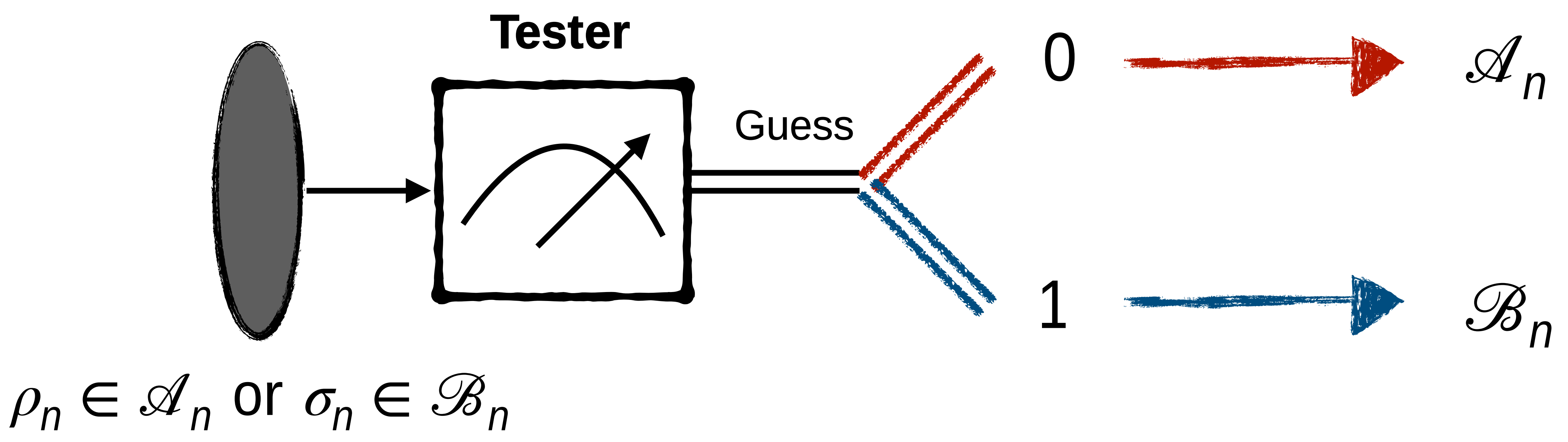
Stability assumption $$\leq \lim_{n\to\infty} -\frac{1}{mn}\log \alpha_{mn,r}(\rho_m^{\otimes n},\sigma_m^{\otimes n})$$

Hoeffding for i.i.d. states $$= \frac{1}{m}H_{m,r}(\rho_m\|\sigma_m) \longrightarrow \frac{1}{m}H_{m,r}(\mathscr{A}_m\|\mathscr{B}_m) \longrightarrow H_r^{\infty}(\mathscr{A}\|\mathscr{B})$$

To show the limit superior for the upper bound, it requires Nussbaum-Szkoła distributions and the Gartner-Ellis theorem; refer to the full paper arXiv: 2508.12901.

# (III) Strong converse exponent (2508.12901)




$$\alpha_{n,r}(\rho_n \| \sigma_n) := \min_{0 \le M_n \le I} \{\alpha(\rho_n, M_n) : \beta(\sigma_n, M_n) \le 2^{-nr}\}$$

$$\alpha_{n,r}(\mathscr{A}_n \| \mathscr{B}_n) := \min_{0 \le M_n \le I} \{\alpha(\mathscr{A}_n, M_n) : \beta(\mathscr{B}_n, M_n) \le 2^{-nr}\}$$

$$\lim_{n \to \infty} -\frac{1}{n} \log(1 - \alpha_{n,r}(\rho^{\otimes n}, \sigma^{\otimes n})) = H_r^*(\rho \| \sigma)$$

$$\lim_{n \to \infty} -\frac{1}{n} \log(1 - \alpha_{n,r}(\mathscr{A}_n, \mathscr{B}_n)) \stackrel{?}{=} H_r^*(\mathscr{A} \| \mathscr{B})$$

**Quantum Hoeffding anti-divergence**

$$H_r^*(\rho \| \sigma) := \sup_{\alpha > 1} \frac{\alpha - 1}{\alpha} (r - D_{S,\alpha}(\rho \| \sigma))$$

**Sandwiched** $D_{S,\alpha}(\rho \| \sigma) := \frac{1}{\alpha - 1} \log \mathrm{Tr} \left[\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}}\right]^{\alpha}$

**How to define $H_r^*(\mathscr{A} \| \mathscr{B})$ ?**

# (III) Strong converse exponent (2508.12901)

## How to define $H_r^*(\mathscr{A}\|\mathscr{B})$ ?

$$D_{S,\alpha}(\mathscr{A}_n\|\mathscr{B}_n) := \inf_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} D_{S,\alpha}(\rho_n\|\sigma_n)$$

$$H^*_{n,r}(\rho_n\|\sigma_n) := \sup_{\alpha>1} \frac{\alpha-1}{\alpha}\left(nr - D_{S,\alpha}(\rho_n\|\sigma_n)\right)$$

Extension 1

$$H^*_{n,r}(\mathscr{A}_n\|\ \mathscr{B}_n) := \sup_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} H^*_{n,r}(\rho_n\|\sigma_n)$$

Extension 2

$$\mathfrak{H}^*_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\alpha>1} \frac{\alpha-1}{\alpha}\left(nr - D_{S,\alpha}(\mathscr{A}_n\|\mathscr{B}_n)\right)$$

**Are they equivalent?**

$$\sup_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} \sup_{\alpha>1} \frac{\alpha-1}{\alpha}\left(nr - D_{S,\alpha}(\rho_n\|\sigma_n)\right) = \sup_{\alpha>1} \sup_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} \frac{\alpha-1}{\alpha}\left(nr - D_{S,\alpha}(\rho_n\|\sigma_n)\right)$$

No minimax issue here

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$$H_{n,r}^*(\mathscr{A}_n\|\ \mathscr{B}_n) := \sup_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} H_{n,r}^*(\rho_n\|\sigma_n)$$

Extension 2

$$\mathfrak{H}_{n,r}^*(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\alpha>1}\frac{\alpha-1}{\alpha}\left(nr - D_{\mathrm{S},\alpha}(\mathscr{A}_n\|\mathscr{B}_n)\right)$$

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$$H_{n,r}^*(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} H_{n,r}^*(\rho_n\|\sigma_n)$$

$$H_r^{*,\infty}(\mathscr{A}\|\mathscr{B}) := \lim_{n\to\infty} \frac{1}{n} H_{n,r}^*(\mathscr{A}_n\|\mathscr{B}_n)$$

$$\mathfrak{H}_{n,r}^*(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\alpha>1} \frac{\alpha-1}{\alpha}\left(nr - D_{\mathrm{S},\alpha}(\mathscr{A}_n\|\mathscr{B}_n)\right)$$

$$\mathfrak{H}_r^{*,\infty}(\mathscr{A}\|\mathscr{B}) := \sup_{\alpha>1} \frac{\alpha-1}{\alpha}\left(r - D_{\mathrm{S},\alpha}^{\infty}(\mathscr{A}\|\mathscr{B})\right)$$

## Are they equivalent?

$$\lim_{n\to\infty} \sup_{\alpha>1} \frac{\alpha-1}{\alpha}\left(r - \frac{1}{n} D_{\mathrm{S},\alpha}(\mathscr{A}_n\|\mathscr{B}_n)\right)$$

$$\lim_{n\to\infty} \Updownarrow \sup_{n>1}$$

Superadditivity in $n$

$$\sup_{\alpha>1} \lim_{n\to\infty} \frac{\alpha-1}{\alpha}\left(r - \frac{1}{n} D_{\mathrm{S},\alpha}(\mathscr{A}_n\|\mathscr{B}_n)\right)$$

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$$H_{n,r}^*(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} H_{n,r}^*(\rho_n\|\sigma_n)$$

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**Are they equivalent?**

$$\underbrace{\lim_{n\to\infty}}_{\Updownarrow\ \sup_{n>1}} \underbrace{\sup_{\alpha>1}\frac{\alpha-1}{\alpha}\left(r-\frac{1}{n}D_{\mathrm{S},\alpha}(\mathscr{A}_n\|\mathscr{B}_n)\right)}_{\text{Superadditivity in } n} = \sup_{\alpha>1}\underbrace{\lim_{n\to\infty}}_{\Updownarrow\ \sup_{n>1}}\underbrace{\frac{\alpha-1}{\alpha}\left(r-\frac{1}{n}D_{\mathrm{S},\alpha}(\mathscr{A}_n\|\mathscr{B}_n)\right)}_{\text{Superadditivity in } n}$$

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## How to define $H_r^*(\mathscr{A}\|\mathscr{B})$ ?

$$D_{\mathrm{S},\alpha}(\mathscr{A}_n\|\mathscr{B}_n) := \inf_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} D_{\mathrm{S},\alpha}(\rho_n\|\sigma_n)$$

$$H^*_{n,r}(\rho_n\|\sigma_n) := \sup_{\alpha>1}\frac{\alpha-1}{\alpha}\left(nr - D_{\mathrm{S},\alpha}(\rho_n\|\sigma_n)\right)$$

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$$H^*_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} H^*_{n,r}(\rho_n\|\sigma_n)$$

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$$H^{*,\infty}_r(\mathscr{A}\|\mathscr{B}) := \lim_{n\to\infty}\frac{1}{n}H^*_{n,r}(\mathscr{A}_n\|\mathscr{B}_n)$$

$$\mathfrak{H}^{*,\infty}_r(\mathscr{A}\|\mathscr{B}) := \sup_{\alpha>1}\frac{\alpha-1}{\alpha}\left(r - D^{\infty}_{\mathrm{S},\alpha}(\mathscr{A}\|\mathscr{B})\right)$$

Strong converse exponent $\displaystyle \lim_{n\to\infty} -\frac{1}{n}\log(1-\alpha_{n,r}(\mathscr{A}_n\|\mathscr{B}_n))$

The dashed arrow indicates partial progress when $\mathscr{A}_n$ is a singleton.

# Refinement of the Stein exponent regime

**Error exponent**

$$\liminf_{n\to\infty} -\frac{1}{n}\log \alpha_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) \geq \mathfrak{H}_r^{\infty}(\mathscr{A}\|\mathscr{B})$$

**Strong converse exponent**

$$\liminf_{n\to\infty} -\frac{1}{n}\log(1-\alpha_{n,r}(\mathscr{A}_n\|\mathscr{B}_n)) \geq \mathfrak{H}_r^{*,\infty}(\mathscr{A}\|\mathscr{B})$$

$$\mathfrak{H}_r^{\infty}(\mathscr{A}\|\mathscr{B}) := \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\left(r - D_{\mathrm{P},\alpha}^{\infty}(\mathscr{A}\|\mathscr{B})\right)$$

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$$\mathfrak{H}_r^{*,\infty}(\mathscr{A}\|\mathscr{B}) := \sup_{\alpha>1} \frac{\alpha-1}{\alpha}\left(r - D_{\mathrm{S},\alpha}^{\infty}(\mathscr{A}\|\mathscr{B})\right)$$

Using the same assumptions in **[KF-Fawzi-Fawzi-24]**

$$\sup_{\alpha\in(0,1)} D_{\mathrm{P},\alpha}^{\infty}(\mathscr{A}\|\mathscr{B}) = D^{\infty}(\mathscr{A}\|\mathscr{B}) = \inf_{\alpha>1} D_{\mathrm{S},\alpha}^{\infty}(\mathscr{A}\|\mathscr{B})$$

# Refinement of the Stein exponent regime

**Error exponent**

$$\liminf_{n\to\infty} -\frac{1}{n}\log \alpha_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) \geq \mathfrak{H}_r^{\infty}(\mathscr{A}\|\mathscr{B}) > 0, \quad \forall\, 0 < r < D^{\infty}(\mathscr{A}\|\mathscr{B})$$

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$$\mathfrak{H}_r^{*,\infty}(\mathscr{A}\|\mathscr{B}) := \sup_{\alpha>1} \frac{\alpha-1}{\alpha}\left(r - D_{\mathrm{S},\alpha}^{\infty}(\mathscr{A}\|\mathscr{B})\right)$$

Using the same assumptions in **[KF-Fawzi-Fawzi-24]**

$$\sup_{\alpha\in(0,1)} D_{\mathrm{P},\alpha}^{\infty}(\mathscr{A}\|\mathscr{B}) = D^{\infty}(\mathscr{A}\|\mathscr{B}) = \inf_{\alpha>1} D_{\mathrm{S},\alpha}^{\infty}(\mathscr{A}\|\mathscr{B})$$

Any $r$ below $D^{\infty}(\mathscr{A}\|\mathscr{B})$ is achievable, with the type-I error decaying exponentially at least $\mathfrak{H}_r^{\infty}(\mathscr{A}\|\mathscr{B})$.
Any $r$ exceeds $D^{\infty}(\mathscr{A}\|\mathscr{B})$, the type-I error inevitably converges to one exponentially fast, at least $\mathfrak{H}_r^{*,\infty}(\mathscr{A}\|\mathscr{B})$.
Thus, $D^{\infty}(\mathscr{A}\|\mathscr{B})$ delineates a **sharp threshold** for the asymptotic trade-off in hypothesis testing.

# Refinement of the Stein exponent regime

**Error exponent**

$$\liminf_{n\to\infty} -\frac{1}{n}\log \alpha_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) \geq \mathfrak{H}_r^{\infty}(\mathscr{A}\|\mathscr{B}) > 0, \quad \forall\, 0 < r < D^{\infty}(\mathscr{A}\|\mathscr{B})$$

**Strong converse exponent**

$$\liminf_{n\to\infty} -\frac{1}{n}\log(1-\alpha_{n,r}(\mathscr{A}_n\|\mathscr{B}_n)) \geq \mathfrak{H}_r^{*,\infty}(\mathscr{A}\|\mathscr{B}) > 0, \quad \forall\, r > D^{\infty}(\mathscr{A}\|\mathscr{B})$$

$$\mathfrak{H}_r^{\infty}(\mathscr{A}\|\mathscr{B}) := \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}\left(r - D_{\mathrm{P},\alpha}^{\infty}(\mathscr{A}\|\mathscr{B})\right)$$

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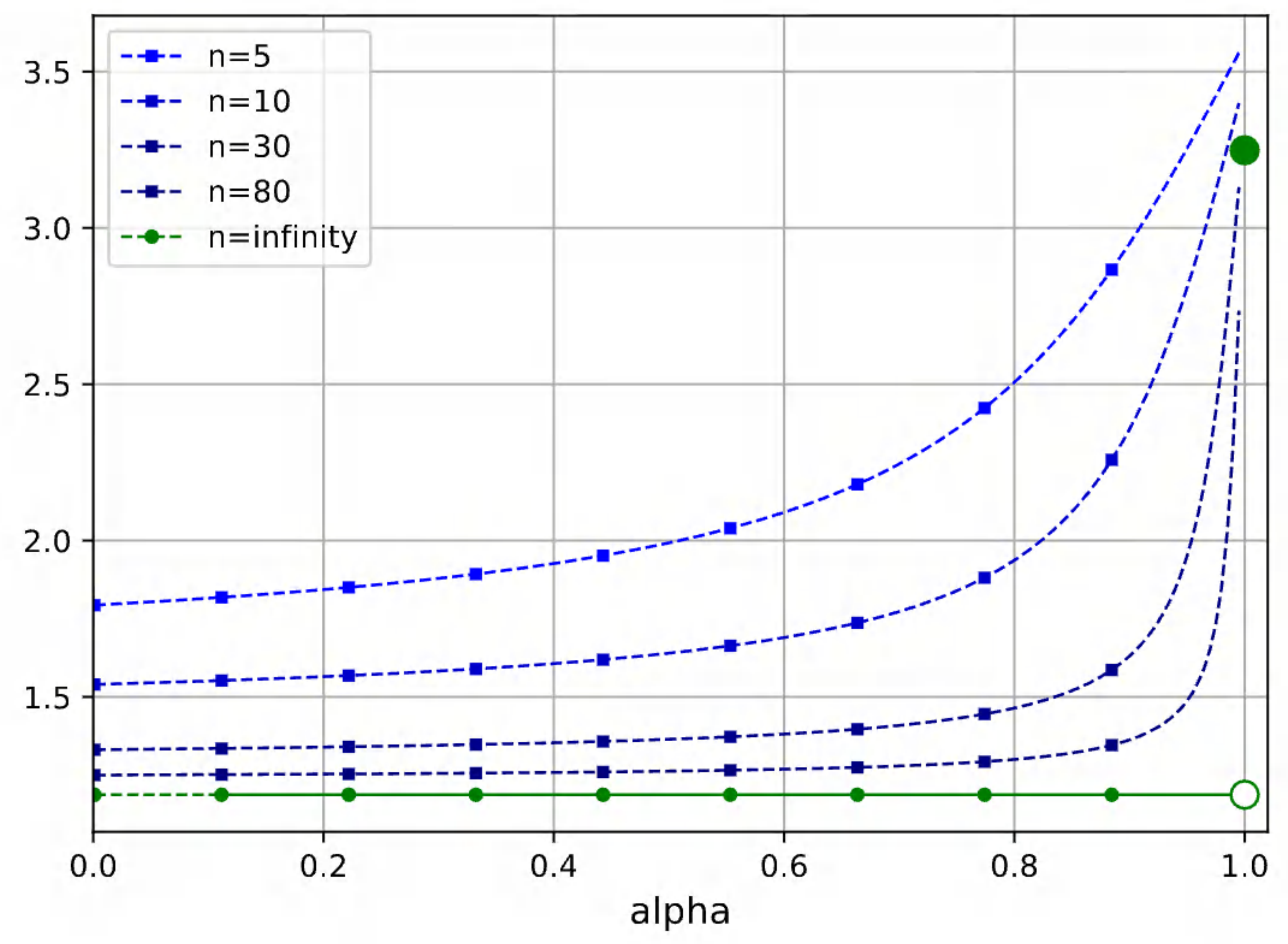
Using the same assumptions in **[KF-Fawzi-Fawzi-24]**

$$\sup_{\alpha\in(0,1)} D_{\mathrm{P},\alpha}^{\infty}(\mathscr{A}\|\mathscr{B}) = D^{\infty}(\mathscr{A}\|\mathscr{B}) = \inf_{\alpha>1} D_{\mathrm{S},\alpha}^{\infty}(\mathscr{A}\|\mathscr{B})$$

**Remark:** Recovering the Stein's setting by [Brandao-Plenio-10], and alternative proof besides [Hayashi-Yamasaki-24; Lami-24], we need $\sup_{\alpha\in(0,1)} D_{\mathrm{P},\alpha}^{\infty}(\mathscr{A}\|\mathscr{B}) = D^{\infty}(\mathscr{A}\|\mathscr{B})$ with the corresponding assumptions.

# Examples for discontinuity of regularized divergence



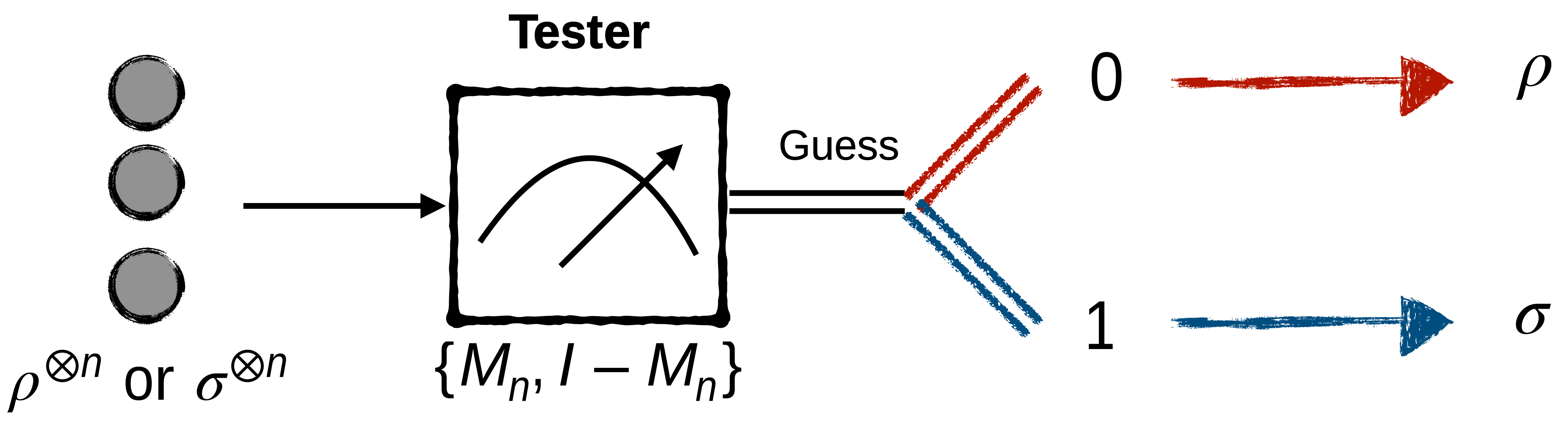

$$\sup_{\alpha\in(0,1)} D_{\mathrm{P},\alpha}(\mathscr{A}_n\|\mathscr{B}_n) = D(\mathscr{A}_n\|\mathscr{B}_n) \quad \forall n \in \mathbb{N}$$

But $$\sup_{\alpha\in(0,1)} D^{\infty}_{\mathrm{P},\alpha}(\mathscr{A}_n\|\mathscr{B}_n) \neq D^{\infty}(\mathscr{A}_n\|\mathscr{B}_n)$$

Error tradeoff for composite correlated hypotheses is much more complicated than the simple i.i.d. case.

**Remark:** Recovering the Stein's setting by [Brandao-Plenio-10], and alternative proof besides [Hayashi-Yamasaki-24; Lami-24], we need $\sup_{\alpha\in(0,1)} D^{\infty}_{\mathrm{P},\alpha}(\mathscr{A}\|\mathscr{B}) = D^{\infty}(\mathscr{A}\|\mathscr{B})$ with the corresponding assumptions.

# (IV) Chernoff exponent (2508.12889)



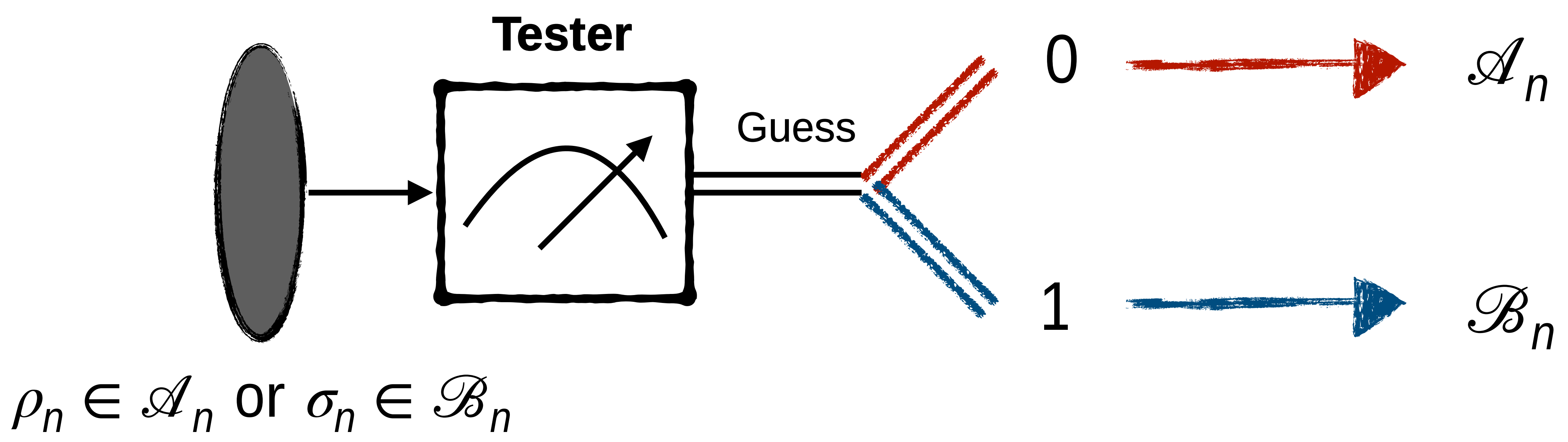

$$P_{e,\min}(\rho_n \| \sigma_n) := \min_{0 \le M_n \le I} \left(\pi_0\, \alpha(\rho_n, M_n) + \pi_1\, \beta(\sigma_n, M_n)\right)$$

$$\lim_{n\to\infty} -\frac{1}{n} \log P_{e,\min}(\rho^{\otimes n}, \sigma^{\otimes n}) = C(\rho\|\sigma)$$

**Quantum Chernoff divergence**

$$C(\rho\|\sigma) := \max_{\alpha\in[0,1]} -\log Q_\alpha(\rho\|\sigma)$$

**Petz-quasi** $\quad Q_\alpha(\rho\|\sigma) := \mathrm{Tr}\,[\rho^\alpha \sigma^{1-\alpha}]$




$$P_{e,\min}(\mathscr{A}_n \| \mathscr{B}_n) := \min_{0 \le M_n \le I} \left(\pi_0\, \alpha(\mathscr{A}_n, M_n) + \pi_1\, \beta(\mathscr{B}_n, M_n)\right)$$

$$\lim_{n\to\infty} -\frac{1}{n} \log P_{e,\min}(\mathscr{A}_n, \mathscr{B}_n) \overset{?}{=} C(\mathscr{A}\|\mathscr{B})$$

**How to define $C(\mathscr{A}\|\mathscr{B})$ ?**

# (IV) Chernoff exponent (2508.12889)

## How to define $C(\mathscr{A}\|\mathscr{B})$ ?

$$Q_\alpha(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} Q_\alpha(\rho_n\|\sigma_n)$$

$$C(\rho_n\|\sigma_n) := \max_{\alpha\in[0,1]} -\log Q_\alpha(\rho_n\|\sigma_n)$$

Extension 1

Extension 2

$$C(\mathscr{A}_n\|\ \mathscr{B}_n) := \inf_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} C(\rho_n\|\sigma_n)$$

$$\mathfrak{C}_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \max_{\alpha\in[0,1]} -\log Q_\alpha(\mathscr{A}_n\|\mathscr{B}_n)$$

**Are they equivalent?**

$$\inf_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} \max_{\alpha\in[0,1]} -\log Q_\alpha(\rho_n\|\sigma_n) \;=\; \max_{\alpha\in[0,1]} \inf_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} -\log Q_\alpha(\rho_n\|\sigma_n)$$

minimax theorem

Concave in $(\rho_n, \sigma_n)$, convex in $\alpha$

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$$Q_\alpha(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} Q_\alpha(\rho_n\|\sigma_n)$$

$$C(\rho_n\|\sigma_n) := \max_{\alpha\in[0,1]} -\log Q_\alpha(\rho_n\|\sigma_n)$$

Extension 1

Extension 2

$$C(\mathscr{A}_n\| \mathscr{B}_n) := \inf_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} C(\rho_n\|\sigma_n) \longleftrightarrow \mathfrak{C}_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \max_{\alpha\in[0,1]} -\log Q_\alpha(\mathscr{A}_n\|\mathscr{B}_n)$$

# (IV) Chernoff exponent (2508.12889)

## How to define $C(\mathscr{A}\|\mathscr{B})$ ?

$$Q_\alpha(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\rho_n\in\mathscr{A}_n,\ \sigma_n\in\mathscr{B}_n} Q_\alpha(\rho_n\|\sigma_n)$$

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$$C^\infty(\mathscr{A}\|\mathscr{B}) := \lim_{n\to\infty}\frac{1}{n}C(\mathscr{A}_n\|\mathscr{B}_n)$$

Extension 2

$$\mathfrak{C}_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \max_{\alpha\in[0,1]} -\log Q_\alpha(\mathscr{A}_n\|\mathscr{B}_n)$$

$$\mathfrak{C}^\infty(\mathscr{A}\|\mathscr{B}) := \max_{\alpha\in[0,1]} -\log Q_\alpha^\infty(\mathscr{A}_n\|\mathscr{B}_n)$$

**Are they equivalent?**

$$\inf_{n\geq 1}\ \max_{\alpha\in[0,1]} -\frac{1}{n}\log Q_\alpha(\mathscr{A}_n\|\mathscr{B}_n) = \max_{\alpha\in[0,1]}\ \inf_{n\geq 1} -\frac{1}{n}\log Q_\alpha(\mathscr{A}_n\|\mathscr{B}_n)$$

minimax theorem
[Mosonyi-Hiai-11]

Upper semicontinuous in $\alpha$, monotone decreasing in $k$, $n = 2^k$

# (IV) Chernoff exponent (2508.12889)

## How to define $C(\mathscr{A}\|\mathscr{B})$ ?

$$Q_\alpha(\mathscr{A}_n\|\mathscr{B}_n) := \sup_{\rho_n\in\mathscr{A}_n,\, \sigma_n\in\mathscr{B}_n} Q_\alpha(\rho_n\|\sigma_n)$$

$$C(\rho_n\|\sigma_n) := \max_{\alpha\in[0,1]} -\log Q_\alpha(\rho_n\|\sigma_n)$$

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$$C(\mathscr{A}_n\| \mathscr{B}_n) := \inf_{\rho_n\in\mathscr{A}_n,\, \sigma_n\in\mathscr{B}_n} C(\rho_n\|\sigma_n)$$

$$C^\infty(\mathscr{A}\|\mathscr{B}) := \lim_{n\to\infty} \frac{1}{n} C(\mathscr{A}_n\|\mathscr{B}_n)$$

Extension 2

$$\mathfrak{C}_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \max_{\alpha\in[0,1]} -\log Q_\alpha(\mathscr{A}_n\|\mathscr{B}_n)$$

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$$C(\rho_n\|\sigma_n) := \max_{\alpha\in[0,1]} -\log Q_\alpha(\rho_n\|\sigma_n)$$

Extension 1

Extension 2

$$C(\mathscr{A}_n\|\mathscr{B}_n) := \inf_{\rho_n\in\mathscr{A}_n,\,\sigma_n\in\mathscr{B}_n} C(\rho_n\|\sigma_n) \longleftrightarrow \mathfrak{C}_{n,r}(\mathscr{A}_n\|\mathscr{B}_n) := \max_{\alpha\in[0,1]} -\log Q_\alpha(\mathscr{A}_n\|\mathscr{B}_n)$$

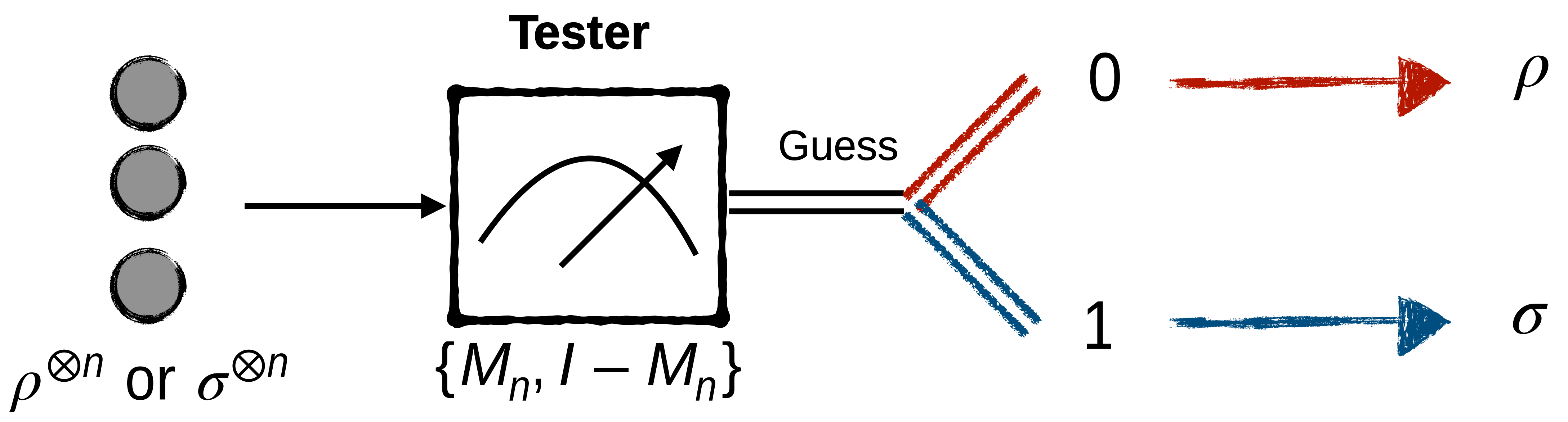
$$C^\infty(\mathscr{A}\|\mathscr{B}) := \lim_{n\to\infty}\frac{1}{n}C(\mathscr{A}_n\|\mathscr{B}_n) \longleftrightarrow \mathfrak{C}^\infty(\mathscr{A}\|\mathscr{B}) := \max_{\alpha\in[0,1]} -\log Q_\alpha^\infty(\mathscr{A}_n\|\mathscr{B}_n)$$

$\updownarrow$

Chernoff exponent $\quad \lim_{n\to\infty} -\frac{1}{n}\log P_{e,\min}(\mathscr{A}_n\|\mathscr{B}_n)$

Same result holds for **multiple hypotheses.** This extends the result by [Li-16]. Refer to the full paper on arXiv: 2508.12889.

# (IV) Chernoff exponent (2508.12889)

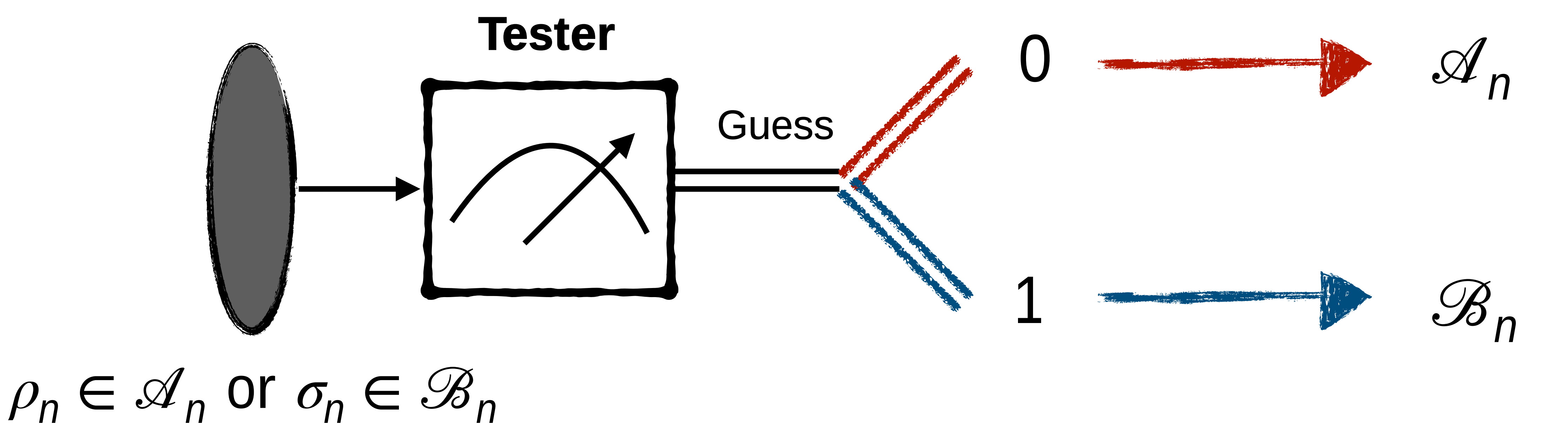



$$P_{e,\min}(\rho_n \| \sigma_n) := \min_{0 \le M_n \le I} \left(\pi_0\, \alpha(\rho_n, M_n) + \pi_1\, \beta(\sigma_n, M_n)\right)$$

$$= \frac{1}{2}\left(1 - \|\pi_0 \rho_n - \pi_1 \sigma_n\|_1\right)$$

$M_n^* = \{\pi_0 \rho_n - \pi_1 \sigma_n \ge 0\}$ Holevo-Helstrom test

Projection onto the positive part of the Hermitian operator




$$P_{e,\min}(\mathscr{A}_n \| \mathscr{B}_n) := \min_{0 \le M_n \le I} \left(\pi_0\, \alpha(\mathscr{A}_n, M_n) + \pi_1\, \beta(\mathscr{B}_n, M_n)\right)$$

**What is the optimal test here?**

# (IV) Chernoff exponent (2508.12889)

$$P_{e,\min}(\mathscr{A}_n \| \mathscr{B}_n) := \min_{0 \le M_n \le I} \underbrace{\left(\pi_0\, \alpha(\mathscr{A}_n, M_n) + \pi_1\, \beta(\mathscr{B}_n, M_n)\right)}_{\text{worst-case error}} = \min_{0 \le M_n \le I} \underline{\sup_{\rho_n \in \mathscr{A}_n, \sigma_n \in \mathscr{B}_n} \left(\pi_0\, \alpha(\rho_n, M_n) + \pi_1\, \beta(\sigma_n, M_n)\right)}$$

$$\min_{0 \le M_n \le I} \boxed{\sup_{\rho_n \in \mathscr{A}_n, \sigma_n \in \mathscr{B}_n} \left(\pi_0\, \alpha(\rho_n, M_n) + \pi_1\, \beta(\sigma_n, M_n)\right)} \overset{\text{minimax}}{=} \sup_{\rho_n \in \mathscr{A}_n, \sigma_n \in \mathscr{B}_n} \boxed{\min_{0 \le M_n \le I} \left(\pi_0\, \alpha(\rho_n, M_n) + \pi_1\, \beta(\sigma_n, M_n)\right)}$$

State-agnostic
(Independent of $\rho_n, \sigma_n$)

Black-box setting

No matter which state you pick, the test should universally suppress the error

State-aware
(dependent of $\rho_n, \sigma_n$)

Pick a pair of states and design a test to discriminate, then optimize all possible pairs of states

$$\sup_{\rho_n \in \mathscr{A}_n, \sigma_n \in \mathscr{B}_n} \frac{1}{2}\left(1 - \|\pi_0 \rho_n - \pi_1 \sigma_n\|_1\right)$$

$$= \frac{1}{2}\left(1 - \|\pi_0 \rho_n^* - \pi_1 \sigma_n^*\|_1\right)$$

The most difficult pair of states are the ones with smallest trace distance

**Is this also optimal for the l.h.s.?**

Optimal test on r.h.s.: $\{\pi_0 \rho_n^* - \pi_1 \sigma_n^* \ge 0\}$

# (IV) Chernoff exponent (2508.12889)

$$P_{e,\min}(\mathscr{A}_n \| \mathscr{B}_n) := \min_{0 \le M_n \le I} \underbrace{\left(\pi_0\, \alpha(\mathscr{A}_n, M_n) + \pi_1\, \beta(\mathscr{B}_n, M_n)\right)}_{\text{worst-case error}} = \min_{0 \le M_n \le I} \underline{\sup_{\rho_n \in \mathscr{A}_n, \sigma_n \in \mathscr{B}_n} \left(\pi_0\, \alpha(\rho_n, M_n) + \pi_1\, \beta(\sigma_n, M_n)\right)}$$

$$\min_{0 \le M_n \le I} \boxed{\sup_{\rho_n \in \mathscr{A}_n, \sigma_n \in \mathscr{B}_n} \left(\pi_0\, \alpha(\rho_n, M_n) + \pi_1\, \beta(\sigma_n, M_n)\right)} \overset{\text{minimax}}{=} \sup_{\rho_n \in \mathscr{A}_n, \sigma_n \in \mathscr{B}_n} \boxed{\min_{0 \le M_n \le I} \left(\pi_0\, \alpha(\rho_n, M_n) + \pi_1\, \beta(\sigma_n, M_n)\right)}$$

**Counter-example:** $\min_{x \in [-1,1]} \max_{y \in [-1,1]} xy = \max_{y \in [-1,1]} \min_{x \in [-1,1]} xy$

**Minimizers on r.h.s. is $[-1,1]$, but the minimizer on l.h.s. is $0$.**

**Is this also optimal for the l.h.s.?**

Optimal test on r.h.s.: $\{\pi_0 \rho_n^* - \pi_1 \sigma_n^* \geq 0\}$

# (IV) Chernoff exponent (2508.12889)

$$P_{e,\min}(\mathscr{A}_n \| \mathscr{B}_n) := \min_{0 \le M_n \le I} \underbrace{\left(\pi_0\, \alpha(\mathscr{A}_n, M_n) + \pi_1\, \beta(\mathscr{B}_n, M_n)\right)}_{\text{worst-case error}} = \min_{0 \le M_n \le I} \underline{\sup_{\rho_n \in \mathscr{A}_n, \sigma_n \in \mathscr{B}_n} \left(\pi_0\, \alpha(\rho_n, M_n) + \pi_1\, \beta(\sigma_n, M_n)\right)}$$

$$\min_{0 \le M_n \le I} \sup_{\rho_n \in \mathscr{A}_n, \sigma_n \in \mathscr{B}_n} \left(\pi_0\, \alpha(\rho_n, M_n) + \pi_1\, \beta(\sigma_n, M_n)\right) \overset{\text{minimax}}{=} \sup_{\rho_n \in \mathscr{A}_n, \sigma_n \in \mathscr{B}_n} \min_{0 \le M_n \le I} \left(\pi_0\, \alpha(\rho_n, M_n) + \pi_1\, \beta(\sigma_n, M_n)\right)$$

**YES, provided that $\pi_0 \rho_n^* - \pi_1 \sigma_n^*$ is full rank.**

**(Minimizer on the r.h.s. is unique. It can be proved as a minimizer on the l.h.s. as well.)**

**Is this also optimal for the l.h.s.?**

Optimal test on r.h.s.: $\{\pi_0 \rho_n^* - \pi_1 \sigma_n^* \ge 0\}$

# (IV) Chernoff exponent (2508.12889)

## Maximum overlap with free states in resource theory

$$O_{\mathscr{F}}(\psi) := \sup_{\sigma \in \mathscr{F}} \langle \psi | \sigma | \psi \rangle$$

A technical quantity that appears frequently in quantum resource theory

[KF-Liu-20, 22; Liu-Bu-Takagi-19; Bravyi et al.-19]

**Resource theory of magic**: $\mathscr{F}$ is the set of stabilizer states and the maximum overlap is closely related to the **stabilizer rank** and **stabilizer extent**—key quantities in fault-tolerant quantum computation.

**Operational meaning:** $C(|\psi\rangle\langle\psi| \| \mathscr{F}) = -\log O_{\mathscr{F}}(\psi)$ Reasonable resource measure

$$O_{\mathrm{STAB}}(\psi_1 \otimes \psi_2) > O_{\mathrm{STAB}}(\psi_1) O_{\mathrm{STAB}}(\psi_2) \longrightarrow C(\psi_1 \otimes \psi_2 \| \mathrm{STAB}) < C(\psi_1 \| \mathrm{STAB}) + C(\psi_2 \| \mathrm{STAB})$$

[Bravyi et al.-19]

Justifying the regularization $C^{\infty}(\mathscr{A} \| \mathscr{B}) := \lim_{n \to \infty} \frac{1}{n} C(\mathscr{A}_n \| \mathscr{B}_n)$

# Summary

★ (A.1) Each $\mathscr{B}_n$ is convex and compact;
(A.3) $\mathscr{B}_m \otimes \mathscr{B}_k \subseteq \mathscr{B}_{m+k}$, for all $m, k \in \mathbb{N}$;

| Operational regimes | Information measure | IID Sources | Composite Correlated |
|---|---|---|---|
| **(I) Stein's exponent** | Quantum relative entropy | [Hiai-Petz-91] [Ogawa-Nagaoka-00] | [Hayashi-Yamasaki-24] [Lami-24] [KF-Fawzi-Fawzi-24] |
| **(II) Error exponent** | Quantum Hoeffding divergence | [Ogawa-Hayashi-02] [Hayashi-06][Nagaoka-06] [Audenaert et al.-07] | **2508.12901 (this talk)** |
| **(III) Strong converse exponent** | Quantum Hoeffding anti-divergence | [Ogawa-Nagaoka-00] [Mosonyi-Ogawa-15] | **2508.12901 (this talk)** |
| **(IV) Chernoff exponent** | Quantum Chernoff divergence | [Audenaert et al.-07 [Nussbaum-Szkoła-09] | **2508.12889 (this talk)** |

**An almost complete picture for composite correlated hypotheses**

(Extremely general; recover the i.i.d. setting as a special case)

Missing part: (III) strong converse exponent for general set $\mathscr{A}_n$

**Applications:** explore the black-box/adversarial/correlated settings for any tasks utilizing the quantum hypothesis testing e.g. channel coding, channel discrimination, data compression, resource theory …

# Thanks for your attention!

**arXiv:2508.12901 & 2508.12889**