Quantum Coherence Manipulation with Finite Resources

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Selected results from **1711.10512** (PRL) **1805.04045** (QUANTUM) + **1804.09500** (PRL)

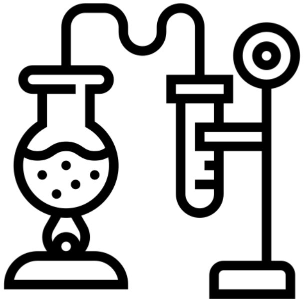
Presented at IQC, University of Waterloo

CENTRE FOR QUANTUM SOFTWARE AND INFORMATION

Talk Outline

- Resource Theory

- Quantum Coherence Distillation
  How many cobits can be distilled?



- Application: Channel Simulation
  How many cobits are required for simulation?

- Summary and Discussions

Resource Theory

Resource theory

Resource Theory = free states + free operations

Entanglement theory = separable states + LOCC

Steering = non-steerable + 1-way LOCC

Athermality = Gibbs states + thermal operations

Coherence theory = incoherent state + incoherent operations

(next slide)

Like different currencies: CAD, EUR, GBP, … trade between …

[Chitambar-Gour-2019] RMP **1806.06107**

Two perspectives of studying:

the asymptotic limits given infinite (i.i.d.) copies of resources ?

tradeoff between param. given finite copies (**one-shot**) of resources ?

Coherence theory

Free states: incoherent (diagonal) states $\mathcal{I} := \{\rho \geq 0 : \mathrm{Tr}\,\rho = 1, \rho = \Delta(\rho)\}$

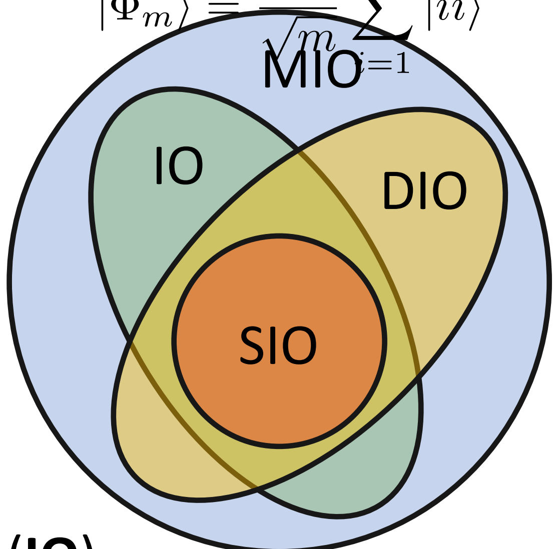
diagonal map

Resource states: coherent (non-diagonal) states

Maximally coherent state: $|\Psi_m\rangle = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} |i\rangle$ Coherent bit (cobit): $|\Psi_2\rangle$

Free operations:

$|\Phi_m\rangle = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} |ii\rangle$ ebit $|\Phi_2\rangle$



Maximally incoherent operations (**MIO**)

$\rho \in \mathcal{I} \implies \mathcal{E}(\rho) \in \mathcal{I}$

$[\Delta \circ \mathcal{E} \circ \Delta = \mathcal{E} \circ \Delta]$

Dephasing-covariant incoherent operations (**DIO**)

$\mathcal{E} \circ \Delta = \Delta \circ \mathcal{E}$

Incoherent operations (**IO**)

$\mathcal{E}(\cdot) = \sum_i E_i \cdot E_i^\dagger,$

$E_i \cdot E_i^\dagger \in \mathrm{MIO}\ \forall i$

Strictly incoherent operations (**SIO**)

$\mathcal{E}(\cdot) = \sum_i E_i \cdot E_i^\dagger,$

$E_i \cdot E_i^\dagger \in \mathrm{DIO}\ \forall i$

Coherence theory

Free states: incoherent (diagonal) states $\mathcal{I} := \{\rho \geq 0 : \operatorname{Tr}\rho = 1, \rho = \Delta(\rho)\}$

Resource states: coherent (non-diagonal) states

Maximally coherent state: $|\Psi_m\rangle = \frac{1}{\sqrt{m}}\sum_{i=1}^{m}|i\rangle$ Coherent bit (cobit): $|\Psi_2\rangle$

Free operations:

Maximally incoherent operations (**MIO**)

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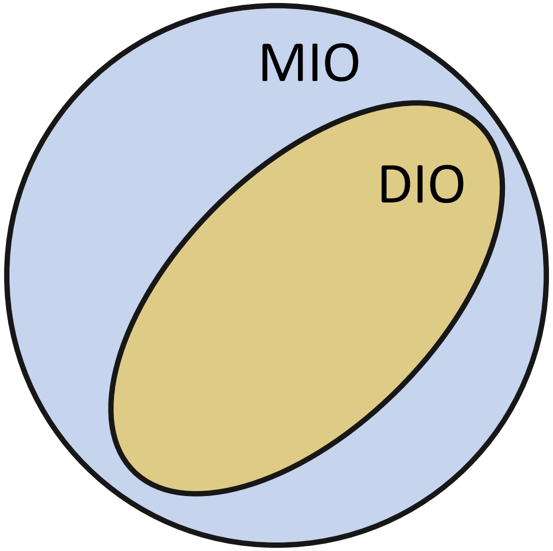


Dephasing-covariant incoherent operations (**DIO**)

$\mathcal{E} \circ \Delta = \Delta \circ \mathcal{E}$

[Streltsov-Adesso-Plenio-2017] RMP **1609.02439**

Why coherence?

Because we can indeed use coherence as a resource in some applications.

This talk: quantum channel simulation.

Other applications of coherence:

PHYSICAL REVIEW A **93**, 012111 (2016)

Coherence as a resource in decision problems: The Deutsch-Jozsa algorithm and a variation

Mark Hillery

PRL **116,** 240405 (2016) PHYSICAL REVIEW LETTERS week ending 17 JUNE 2016

Entanglement and Coherence in Quantum State Merging

A. Streltsov,[1,2,*] E. Chitambar,[3] S. Rana,[1] M. N. Bera,[1] A. Winter,[4,5] and M. Lewenstein[1,5]

Quantum state redistribution with local coherence

Anurag Anshu,[1] Rahul Jain,[2] and Alexander Streltsov[3,4]

Why coherence?

ARTICLE

Received 20 Oct 2015 | Accepted 22 Apr 2016 | Published 20 May 2016 DOI: 10.1038/ncomms11712 OPEN

Numerical approach for unstructured quantum key distribution

Patrick J. Coles[1], Eric M. Metodiev[1] & Norbert Lütkenhaus[1]

It is interesting to point out the connection to coherence[44]. For some set of orthogonal projectors $\Pi=\{\Pi^j\}$ that decompose the identity, $\sum_j \Pi^j=\mathbb{1}$, the coherence (sometimes called relative entropy of coherence) of state ρ is defined as[44]:

$$\Phi(\rho,\Pi)=D\left(\rho\middle\|\sum_j \Pi^j\rho\Pi^j\right). \qquad (42)$$

Rewriting the primal problem in terms of coherence gives

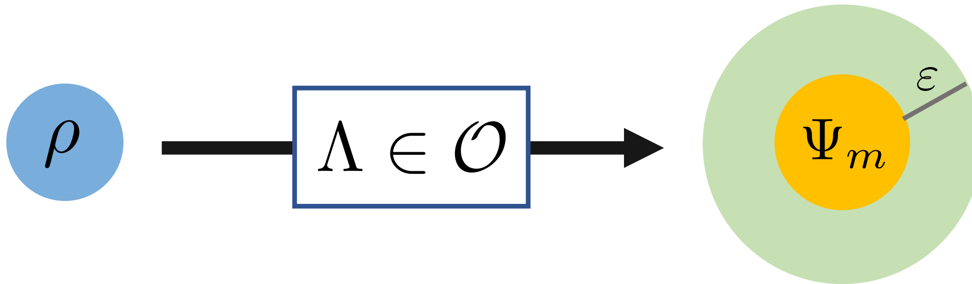
$$\alpha=\min_{\rho_{AB}\in\mathcal{C}}\Phi(\rho_{AB},Z_A). \qquad (43)$$

Hence, we make the connection that calculating the secret key rate is related to optimizing the coherence.

This observation is important since the coherence is a continuous function of ρ (Supplementary Note 6). This allows us to argue in Supplementary Note 6 that our optimization problem satisfies the strong duality criterion[16], which means that the solution of the dual problem is precisely equal to that of primal problem.

Coherence Distillation

One-shot coherence distillation



One-shot distillable coherence

$$C_{d,\mathcal{O}}^{(1),\varepsilon}(\rho) := \max_{\Lambda\in\mathcal{O}} \log_2 m \quad \text{s.t. } F(\Lambda(\rho), \Psi_m) \geq 1-\varepsilon.$$

The asymptotic distillable coherence

$$C_{d,\mathcal{O}}^{\infty}(\rho) = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}C_{d,\mathcal{O}}^{(1),\varepsilon}(\rho^{\otimes n})$$

Question: can we compute $C_{d,\mathcal{O}}^{(1),\varepsilon}(\rho)$?

Distillation under MIO and DIO

MIO

$$C_{d,\mathrm{MIO}}^{(1),\varepsilon}(\rho) = -\log_2 \min \eta \quad \text{s.t. } \operatorname{Tr} G\rho \geq 1-\varepsilon,\ 0 \leq G \leq \mathbb{1},\ \Delta(G) = \eta \mathbb{1}.$$

$$\min x + y \quad \text{s.t. } 2x + 3y \leq 1,\ x - 5y \geq 4.$$

Semidefinite program (SDP)

- A generalization of linear program (LP) from scalar variables to linear operators
- **Efficiently computable**
  - CVX for MATLAB
  - QI toolbox QETLAB by N. Johnston
- **Duality** (Primal-Dual problems)
- Refer to J. Watrous' lecture note for details

Distillation under MIO and DIO

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Distillation under MIO and DIO

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$$C_{d,\mathrm{MIO}}^{(1),\varepsilon}(\rho) = -\log_2 \min \eta$$
$$\text{s.t. } \operatorname{Tr} G\rho \geq 1-\varepsilon,\ 0 \leq G \leq \mathbb{1},\ \Delta(G) = \eta \mathbb{1}.$$

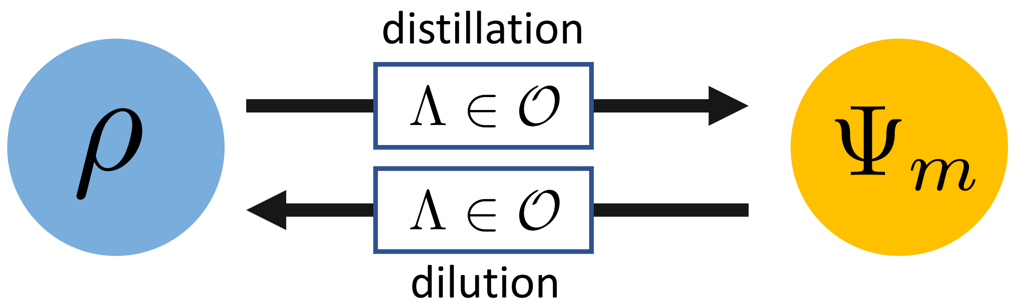
DIO

$$C_{d,\mathrm{DIO}}^{(1),\varepsilon}(\rho) = -\log_2 \min \eta$$
$$\text{s.t. } \operatorname{Tr} G\rho \geq 1-\varepsilon,\ 0 \leq G \leq \mathbb{1},\ \Delta(G) = \eta \mathbb{1}.$$

Q: What is the difference between these two SDPs?

NO difference!

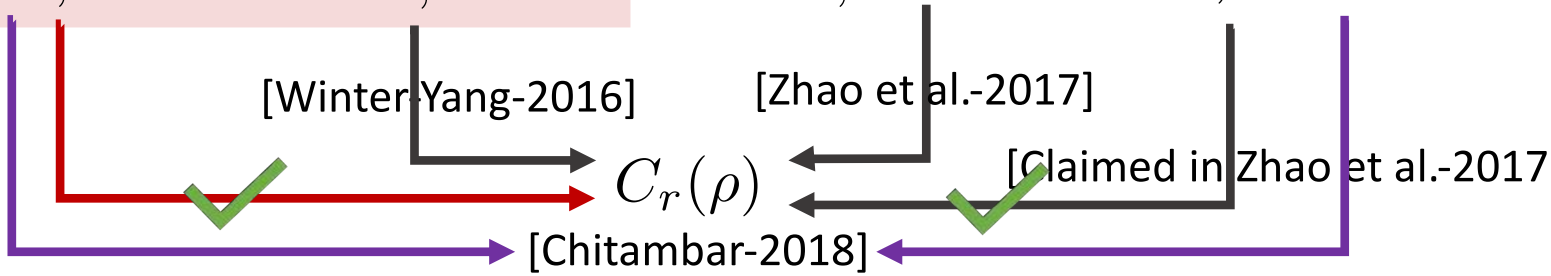
Asymptotic reversibility



$$C^{\infty}_{d,\mathcal{O}}(\rho) \ \overset{?}{\geq} \ C^{\infty}_{c,\mathcal{O}}(\rho)$$

$$C^{(1),\varepsilon}_{d,\mathrm{DIO}}(\rho) = C^{(1),\varepsilon}_{d,\mathrm{MIO}}(\rho) \qquad C_r(\rho) := D(\rho \| \Delta(\rho))$$

$$C^{\infty}_{d,\mathrm{DIO}}(\rho) = C^{\infty}_{d,\mathrm{MIO}}(\rho) = C^{\infty}_{c,\mathrm{MIO}}(\rho) = C^{\infty}_{c,\mathrm{DIO}}(\rho)$$



Reversibility for entanglement theory [Brandão-Plenio-2010] and other resource theories [Brandão-Gour-2015] only known under **resource (asymptotically) non-generating maps**.

Hypothesis testing characterization

$\mathcal{O} \in \{\text{MIO}, \text{DIO}\}$

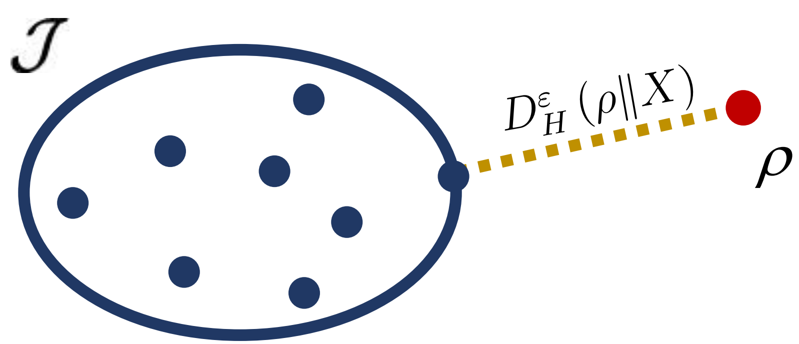
$$C_{d,\mathcal{O}}^{(1),\varepsilon}(\rho) = -\log_2 \min \eta \quad \text{s.t.}\ \operatorname{Tr} G\rho \geq 1-\varepsilon,\ 0 \leq G \leq \mathbb{1},\ \Delta(G) = \eta \mathbb{1}.$$

What does this SDP mean?

Hypothesis testing relative entropy:

$$D_H^{\varepsilon}(\rho\|\sigma) := -\log_2 \min \operatorname{Tr} G\rho \quad \text{s.t.}\ \operatorname{Tr} G\sigma \geq 1-\varepsilon,\ 0 \leq G \leq \mathbb{1}.$$

$C_{d,\text{MIO}}^{(1),\varepsilon}(\rho) = \min_{X \in \mathcal{J}} D_H^{\varepsilon}(\rho\|X)$ with $\mathcal{J} = \{X : \operatorname{Tr} X = 1, \Delta(X) = X\}$



Not necessarily positive semidefinite!

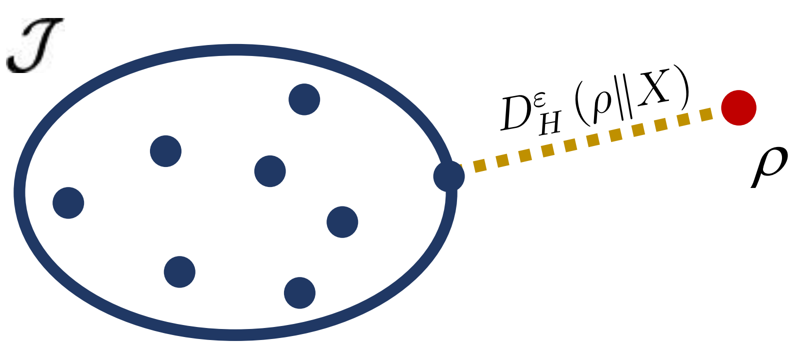
Comparison

Compare with other similar results e.g., [Zhao-Liu-Yuan-Chitambar-Winter-2019]

$$\min_{\sigma\in\mathcal{I}} D_{\min}^{\frac{\varepsilon}{2}-\eta}(\rho\|\sigma) + 2\log\eta \le C_{d,\mathrm{IO}}^{(1),\varepsilon}(\rho) \le \min_{\sigma\in\mathcal{I}} D_{\min}^{\sqrt{\varepsilon(2-\varepsilon)}}(\rho\|\sigma)$$

bounds + correction terms

$C_{d,\mathrm{MIO}}^{(1),\varepsilon}(\rho) = \min_{X\in\mathcal{J}} D_H^{\varepsilon}(\rho\|X)$ with $\mathcal{J} = \{X : \mathrm{Tr}\,X = 1, \Delta(X) = X\}$

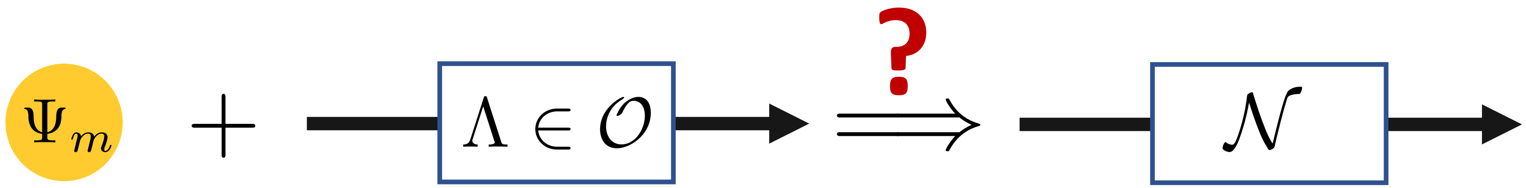


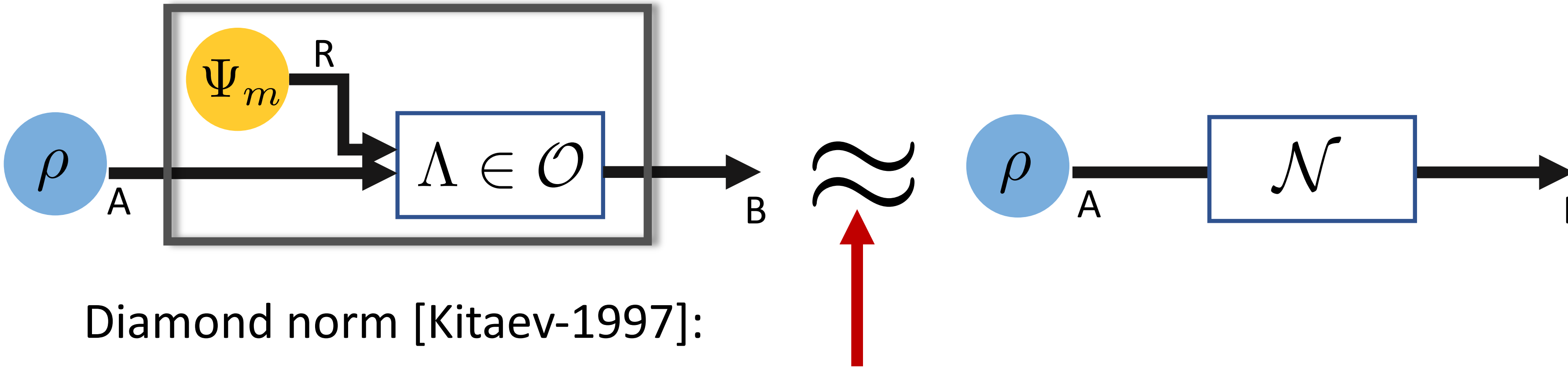
Not necessarily positive semidefinite!

weird + clean

Channel Simulation via Quantum Coherence

Channel simulation setting





Diamond norm [Kitaev-1997]:

$$\|\mathcal{N}_{A\to B} - \mathcal{M}_{A\to B}\|_\diamond := \sup_{\rho_{RA}} \|\mathcal{N}_{A\to B}(\rho_{RA}) - \mathcal{M}_{A\to B}(\rho_{RA})\|_1$$

One-shot simulation cost

$$S_{c,\mathcal{O}}^{(1),\varepsilon} := \min_{\Lambda\in\mathcal{O}} \log_2 m \quad \text{s.t. } \frac{1}{2}\|\Lambda_{AR\to B}(\Psi_m \otimes \cdot) - \mathcal{N}_{A\to B}(\cdot)\|_\diamond \le \varepsilon$$

Question: can we compute $S_{c,\mathcal{O}}^{(1),\varepsilon}$?

SDP for simulation cost

SDP for diamond norm [Watrous-2009]:

$$\frac{1}{2}\|\mathcal{N}_1 - \mathcal{N}_2\|_\diamond = \min \gamma$$
$$\text{s.t. } \operatorname{Tr}_B Y_{AB} \le \gamma \cdot \mathbb{1}_A,$$
$$Y_{AB} \ge J_{\mathcal{N}_1} - J_{\mathcal{N}_2},\ Y_{AB} \ge 0.$$

$$S_{c,\mathrm{MIO}}^{(1),\varepsilon}(\mathcal{N}) = \log_2 \min\ \operatorname{Tr} J_{\widetilde{\mathcal{M}}} \,/\, d_A$$
$$\text{s.t. } 0 \le J_{\widetilde{\mathcal{N}}} \le J_{\widetilde{\mathcal{M}}},$$
$$\operatorname{Tr}_B J_{\widetilde{\mathcal{N}}} = \mathbb{1}_A,$$
$$\operatorname{Tr}_B J_{\widetilde{\mathcal{M}}} = \operatorname{Tr} J_{\widetilde{\mathcal{M}}} \,/\, d_A \cdot \mathbb{1}_A,$$
$$\operatorname{Tr}_A J_{\widetilde{\mathcal{M}}} (|i\rangle\langle i| \otimes \mathbb{1}_B) \in \Delta,\ \forall i$$
$$\operatorname{Tr}_B Y_{AB} \le \varepsilon \cdot \mathbb{1}_A$$
$$Y_{AB} \ge J_{\widetilde{\mathcal{N}}} - J_{\mathcal{N}},\ Y_{AB} \ge 0.$$

Does not look nice but indeed an SDP!

The proof technique does not work for DIO. $\Lambda_{RA\to B}$

Distance characterization

The one-shot coherence simulation cost under MIO is given by

$$S_{c,\mathrm{MIO}}^{(1),\varepsilon}(\mathcal{N}) = \min_{\mathcal{M}\in\mathrm{MIO}} D_{\max}^{\varepsilon}(\mathcal{N}\|\mathcal{M}),$$

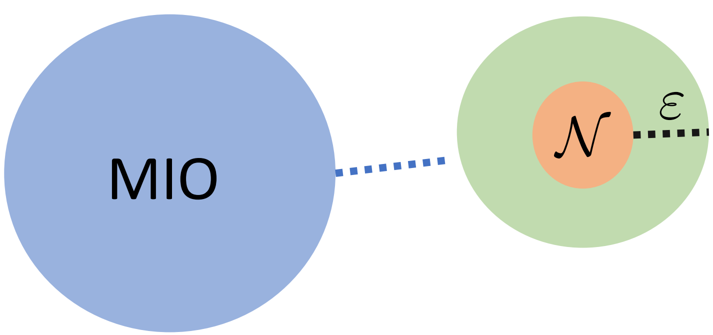
with the channel divergence

$$D_{\max}(\mathcal{N}\|\mathcal{M}) := D_{\max}(J_{\mathcal{N}}\|J_{\mathcal{M}}) \quad \text{and} \quad D_{\max}^{\varepsilon}(\mathcal{N}\|\mathcal{M}) := \inf_{\substack{\frac{1}{2}\|\widehat{\mathcal{N}}-\mathcal{N}\|_{\diamond}\leq\varepsilon \\ \widehat{\mathcal{N}}\in\mathrm{CPTP}}} D_{\max}(\widehat{\mathcal{N}}\|\mathcal{M}).$$

Compatible with other channel divergence

[Cooney-Mosonyi-Wilde-2016; Leditzky-Kaur-Datta-Wilde-2018]

$$\mathbf{D}(\mathcal{N}\|\mathcal{M}) := \max_{|\varphi\rangle_{RA}} \mathbf{D}(\mathcal{N}_{A\to B}(\varphi_{RA})\|\mathcal{M}_{A\to B}(\varphi_{RA}))$$



Channel resource theory

[KF-Wang-Tomomichel-Berta-2018]	1807.05354
[Faist-Berta-Brandão-2018]	1807.05610
[Gour-Wilde-2018]	1808.06980
[Gour-2018]	1808.02607
[Li-Bu-Liu-2018]	1812.02572
……	

Summary & Discussions

Summary

- **Quantum coherence distillation**
  How many cobits can be distilled?
  SDP and entropy characterizations for
  one-shot distillable coherence under MIO and DIO
  which have the same power;

- **Application: Channel simulation**
  How many cobits are required for simulation?
  SDP and entropy characterizations for
  the coherence cost of channel simulation under MIO.

Recent progress & Open question 1

One-shot

		MIO	DIO	IO	SIO
One-shot	distillation	$\tilde{C}_H^{\varepsilon}$ [25]	$\tilde{C}_H^{\varepsilon}$ [25]	$\mathbf{C}^{\varepsilon'}_{\min}$ [*]	**Thm. 10** [*]
	formation	$C^{\varepsilon}_{\max}$ [24]	$C^{\varepsilon}_{\Delta,\max}$ [24]	C^{ε}_{0} [24]	C^{ε}_{0} [24]

One-Shot Coherence Distillation: Towards Completing the Picture

[Zhao-Liu-Yuan-Chitambar-Winter-2018]

What about n-shot scenario? $\rho^{\otimes n}$

Size of $\rho^{\otimes n}$ increases exponentially fast w.r.t. n

Recent progress & Open question 1

Asymptotics

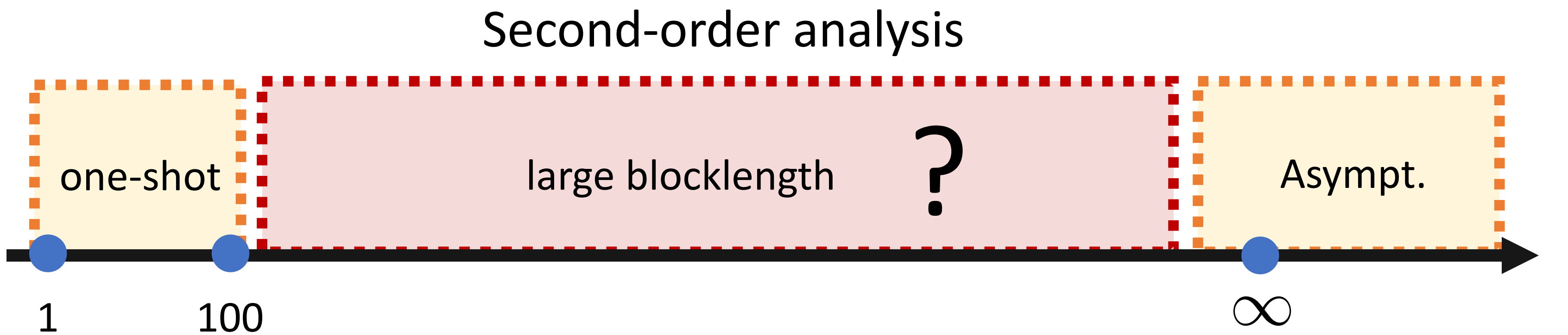
Operations	Distillable coherence	Coherence cost
MIO	C_r	C_r
DIO	C_r	C_r
IO	C_r	C_f
SIO	Q	C_f
PIO	Q	$C_f^{\mathcal{U}}$

Completing the Grand Tour of Asymptotic Quantum Coherence Manipulation [Lami-2019]

Recent progress & Open question 1

For example,

$$C^{(1),\varepsilon}_{d,\mathrm{MIO}}(\rho^{\otimes n}) \stackrel{?}{=} nD(\rho\|\Delta(\rho)) + \sqrt{nV(\rho\|\Delta(\rho))}\,\Phi^{-1}(\varepsilon) + O(\log n)$$



Recent progress & Open question 2

One-shot simulation cost:

$$S_{c,\mathrm{MIO}}^{(1),\varepsilon}(\mathcal{N}) = \min_{\mathcal{M}\in\mathrm{MIO}} D_{\max}^{\varepsilon}(\mathcal{N}\|\mathcal{M}),$$

Asymptotic simulation cost:

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}S_{c,\mathrm{MIO}}^{(1),\varepsilon}(\mathcal{N}^{\otimes n}) = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\min_{\mathcal{M}^n\in\mathrm{MIO}} D_{\max}^{\varepsilon}(\mathcal{N}^{\otimes n}\|\mathcal{M}^n) \longrightarrow ?$$

Partial progress: for classical-quantum channels

$$\longrightarrow \max_{\rho} C_r(\mathcal{N}(\rho)), \quad \text{where} \quad C_r(\rho) := D(\rho\|\Delta(\rho))$$

More about me

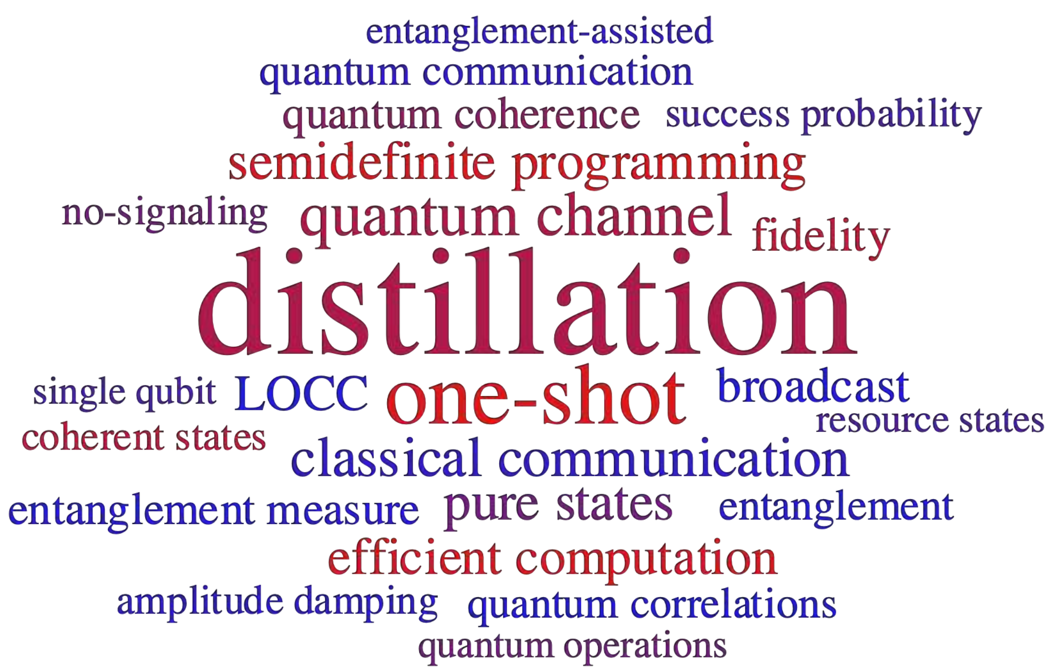
Webpage: kunfang.info

Also work on:

- ◎ entanglement theory
- ◎ channel capacity

Recently working on:

- ◎ polynomial optimization
- ◎ sum-of-square hierarchies
- ◎ uncertainty relations



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More about me

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Happy to discuss any QUANTUM problems

with or without optimizations!

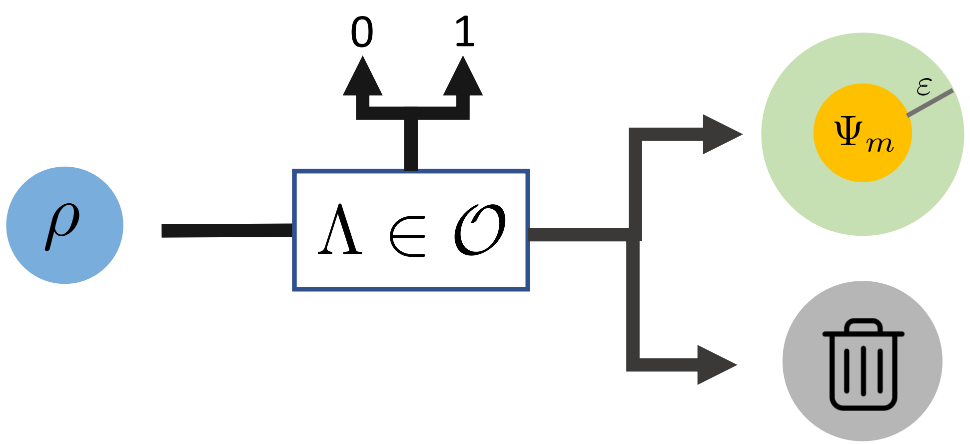
[IQC office: QNC 0213]

Thanks for your attention!

This talk: 1711.10512 (PRL) **1805.04045** (QUANTUM) + **1804.09500** (PRL)

Something more…

Probabilistic distillation



Maximum success probability of coherence distillation:

$$
\begin{aligned}
P_{\mathcal{O}}(\rho \to \Psi_m, \varepsilon) &:= \max p \\
\text{s.t. } & \Lambda_{A\to FB}(\rho) = p|0\rangle\langle 0|_F \otimes \rho' + (1-p)|1\rangle\langle 1|_F \otimes \omega, \\
& F(\rho', \Psi_m) \geq 1 - \varepsilon, \Lambda \in \mathcal{O}.
\end{aligned}
$$

Question: how to compute $P_{\mathcal{O}}(\rho \to \Psi_m, \varepsilon)$?

SDP characterization

The maximal success probability of distillation under MIO/DIO are given by SDPs:

$$
\begin{aligned}
P_{\mathrm{MIO}}(\rho \to \Psi_m, \varepsilon) = \max \;& \operatorname{Tr} G\rho \\
\text{s.t.} \;& \Delta(G) = m\Delta(C), \\
& 0 \le C \le G \le \mathbb{1}, \\
& \operatorname{Tr} C\rho \ge (1-\varepsilon) \operatorname{Tr} G\rho.
\end{aligned}
$$

$$
\begin{aligned}
P_{\mathrm{DIO}}(\rho \to \Psi_m, \varepsilon) = \max \;& \operatorname{Tr} G\rho \\
\text{s.t.} \;& \Delta(G) = m\Delta(C), \\
& 0 \le C \le G \le \mathbb{1}, \\
& \operatorname{Tr} C\rho \ge (1-\varepsilon) \operatorname{Tr} G\rho, \\
& G = \Delta(G).
\end{aligned}
$$

No-go theorem

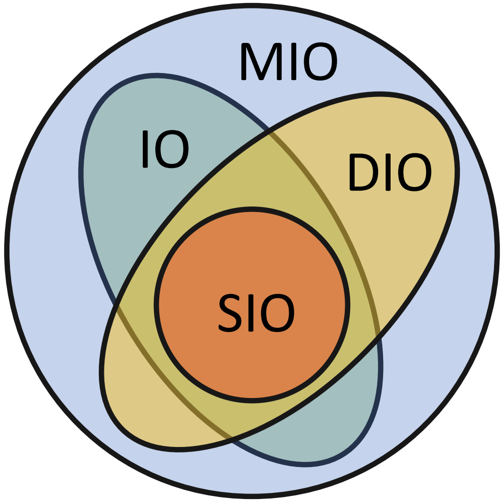
For any full-rank quantum state ρ, it holds that $P_{\mathrm{MIO}}(\rho \to \Psi_m, 0) = 0$.

- MIO sets a fundamental limits for all other incoherent operations;
- Any generic density matrix has full rank;
- Depolarizing noise:

$$\mathcal{D}_\alpha(\rho) = (1-\alpha)\cdot\rho + \alpha\cdot\mathbb{1}/m$$

- Non-continuity: $F(\mathcal{D}_\varepsilon(\Psi_m), \Psi_m) \to 1$

but $|P_{\mathrm{MIO}}(\mathcal{D}_\varepsilon(\Psi_m) \to \Psi_m, 0) - P_{\mathrm{MIO}}(\Psi_m \to \Psi_m, 0)| = 1$



Pure state + MIO

For any pure states $|\varphi\rangle = \sum_{i=1}^{n} \varphi_i |i\rangle,\ \varphi_i \neq 0,\ n \geq 2,$ it holds

$$P_{\mathrm{MIO}}(\varphi \to \Psi_m, 0) \geq \frac{n^2}{m(\sum_{i=1}^{n} |\varphi_i|^{-2})} > 0$$

Use **one cobit** resource to win a **million cobits** lottery! 

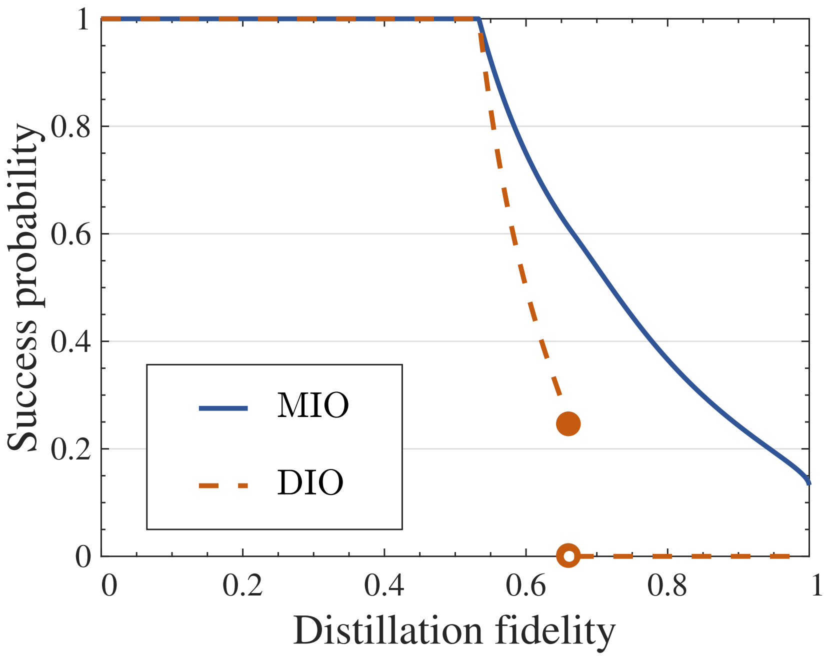
$$P_{\mathrm{MIO}}(\Psi_2 \to \Psi_2^{\otimes 10^6}, 0) \geq \frac{1}{2^{10^6}}.$$

Not happening for DIO!

``Sudden death`` for DIO

For any pure states $|\varphi\rangle = \sum_{i=1}^{n} \varphi_i |i\rangle,\ \varphi_i \neq 0,\ n \geq 2,$ it holds

$$P_{\mathrm{DIO}}(\varphi \to \Psi_m, \varepsilon) \begin{cases} > 0 & \text{if } n \geq m \text{ or if } n < m \text{ and } \varepsilon \geq 1 - \frac{n}{m}, \\ = 0 & \text{if } n < m \text{ and } \varepsilon < 1 - \frac{n}{m}. \end{cases}$$



$$(|0\rangle + 3|1\rangle)/\sqrt{10} \to \Psi_3$$

``Analogous'' to
(pretty) strong converse theorem
in channel coding theory