Quantum Channel Simulation and the Channel's Smooth Max-Information

(1807.05354)

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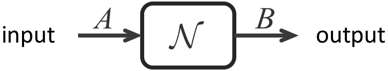
- Background
- Main results
- AEP proof outline
- Summary and discussion

Background

A quantum channel is a communication channel which can transmit quantum information. It sends one quantum state to the other.

Mathematically, a quantum channel is characterized by a linear map $\mathcal{N}_{A\to B}$ that is

- completely positive **(CP)**: $\mathrm{id}_k \otimes \mathcal{N}(X_{RA}) \geq 0 \;\forall\; X_{RA} \geq 0$ and $k \in \mathbb{N}$;
- trace-preserving **(TP)**: $\operatorname{Tr} \mathcal{N}(X) = \operatorname{Tr} X$ for all X.



What is channel simulation?

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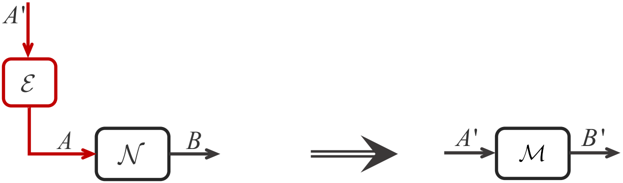

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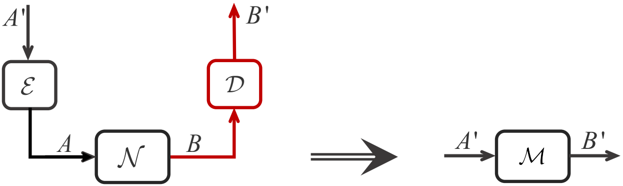

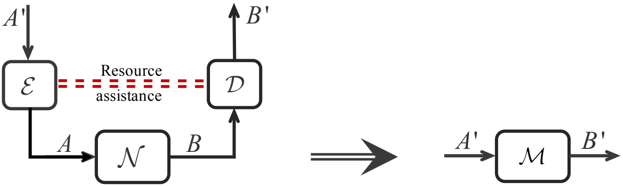
A'
B'
Resource
assistance
A
B
A'
B'

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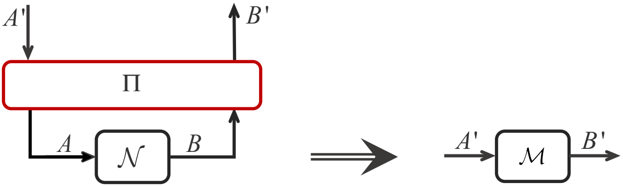

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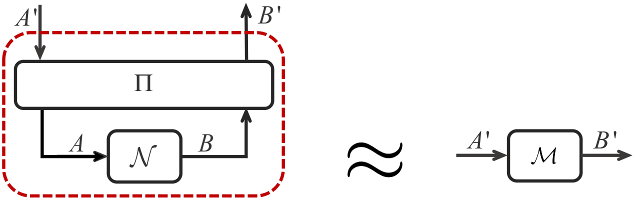

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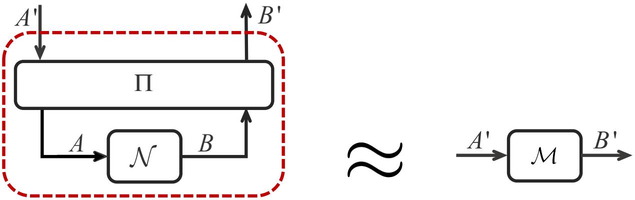

- "Similarity" can be measured via diamond norm [Kitaev, 1997]:

  $\|\mathcal{F}\|_\diamond := \sup_k \|\mathrm{id}_k \otimes \mathcal{F}\|_1$, where $\|\cdot\|_1$induced by the Schatten 1-norm.

- Operational meaning: minimum error probability p_e to distinguish two quantum channels $\mathcal{N}_1$ and $\mathcal{N}_2$ is given by

$$p_e = \frac{1}{2}\left(1 - \frac{\|\mathcal{N}_1 - \mathcal{N}_2\|_\diamond}{2}\right).$$

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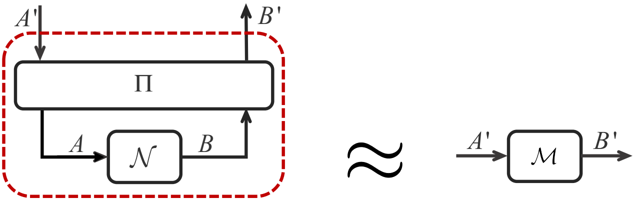

The minimum error of simulation from $\mathcal{N}$ to $\mathcal{M}$ with Ω-assistance:

$$\omega_\Omega(\mathcal{N},\mathcal{M}) := \frac{1}{2}\inf_{\Pi\in\Omega} \|\Pi\circ\mathcal{N}-\mathcal{M}\|_\diamond. \tag{1}$$

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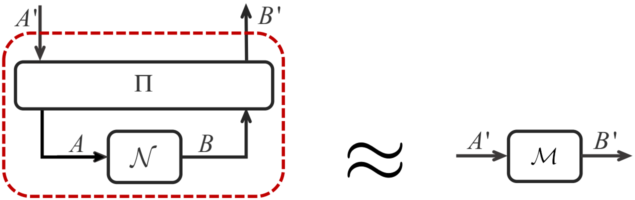

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The channel simulation rate from $\mathcal{N}$ to $\mathcal{M}$ with Ω-assistance:

$$S_\Omega(\mathcal{N},\mathcal{M}) := \lim_{\varepsilon\to 0}\inf\left\{\frac{n}{m} : \omega_\Omega(\mathcal{N}^{\otimes n},\mathcal{M}^{\otimes m}) \le \varepsilon\right\}. \tag{2}$$

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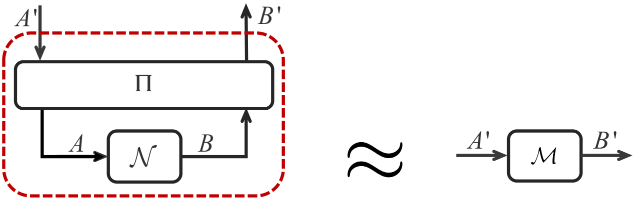

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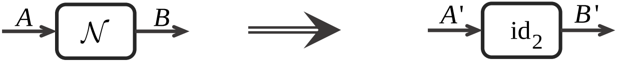
Different resource assistances can be considered. Here we focus on:

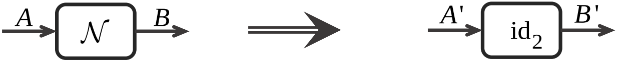
- entanglement assistance, $\Omega = \mathrm{E}$;
- no-signalling (NS) assistance, $\Omega = \mathrm{NS}$;
- $\mathrm{E} \subseteq \mathrm{NS}$.



Q: Optimal rate to simulate the identity channel id_2 via channel $\mathcal{N}$?

In the framework of channel simulation, $Q_\Omega(\mathcal{N}) = S_\Omega(\mathcal{N}, \mathrm{id}_2)^{-1}$.

Capacity vs. simulation cost



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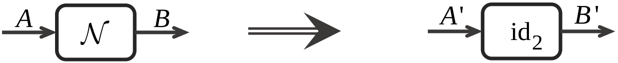

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$$Q_E(\mathcal{N}) \leq Q_{NS}(\mathcal{N}) \leq S_{NS}(\mathcal{N}) \leq S_E(\mathcal{N}). \tag{3}$$

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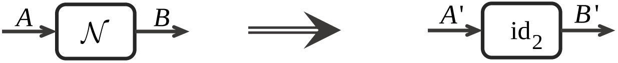

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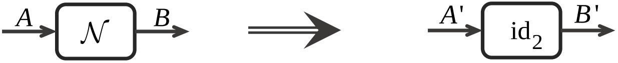

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$$Q_{\mathrm{E}}(\mathcal{N}) \leq Q_{\mathrm{NS}}(\mathcal{N}) \leq S_{\mathrm{NS}}(\mathcal{N}) \leq S_{\mathrm{E}}(\mathcal{N})$$

$$= \frac{1}{2} I(A:B)_{\mathcal{N}} := \frac{1}{2} \max_{\phi_{AA'}} I(A:B)_{\mathcal{N}_{A'\to B}(\phi_{AA'})}$$

↑ [Bennett et al., 2002]

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Quantum reverse Shannon theorem (QRTS)

2017 IEEE Information Theory Society Paper Award

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Main results

Most the prior results focus on the **asymptotic** simulation rate.

Our contributions are twofolds:

- study the **one-shot** channel simulation task: $\mathrm{id}_m \to \mathcal{N}$ with NS-assistance;
- introduce a naturally appeared **entropy of a channel** operational meaning + asymptotic equipartition property (AEP)

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- study the **one-shot** channel simulation task: $\text{id}_m \to \mathcal{N}$ with NS-assistance;
- introduce a naturally appeared **entropy of a channel** operational meaning + asymptotic equipartition property (AEP)

Recall the minimum error of simulation:

$$\omega_{\text{NS}}(\text{id}_m, \mathcal{N}) := \frac{1}{2} \inf_{\Pi \in \text{NS}} \|\Pi \circ \text{id}_m - \mathcal{N}\|_\diamond. \tag{4}$$

The one-shot quantum simulation cost under NS assistance:

$$S^{(1)}_{\text{NS},\varepsilon}(\mathcal{N}) := \log \min \left\{ m \in \mathbb{N} : \omega_{\text{NS}}(\text{id}_m, \mathcal{N}) \leq \varepsilon \right\}. \tag{5}$$

Then the asymptotic quantum simulation cost is equivalently given by

$$S_{\text{NS}}(\mathcal{N}) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} S^{(1)}_{\text{NS},\varepsilon}(\mathcal{N}^{\otimes n}). \tag{6}$$

The minimum error $\omega_{\mathrm{NS}}(\mathrm{id}_m, \mathcal{N})$ can be given by a SDP,

$$\begin{aligned}
\text{minimize} \quad & \lambda && \text{(7a)} \\
\text{subject to} \quad & \operatorname{Tr}_B Y_{AB} \leq \lambda \mathbb{1}_A, && \text{(7b)} \\
& Y_{AB} \geq J_{\widetilde{\mathcal{N}}} - J_{\mathcal{N}}, \; Y_{AB} \geq 0, && \text{(7c)} \\
& J_{\widetilde{\mathcal{N}}} \geq 0, \; \operatorname{Tr}_B J_{\widetilde{\mathcal{N}}} = \mathbb{1}_A, && \text{(7d)} \\
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Sketch of proof: $\omega_{\mathrm{NS}}(\mathrm{id}_m, \mathcal{N}) := \frac{1}{2} \inf_{\Pi \in \mathrm{NS}} \|\Pi \circ \mathrm{id}_m - \mathcal{N}\|_\diamond$,

- [Leung & Matthews, 2015; Duan & Winter, 2016] $\Pi \in \mathrm{NS}$ iff

$$J_\Pi \geq 0,\ \mathrm{Tr}_{AB'}\, J_\Pi = \mathbb{1}_{A'B},$$

$$\mathrm{Tr}_A\, J_\Pi = \mathbb{1}_{A'}/d_{A'} \otimes \mathrm{Tr}_{AA'}\, J_\Pi,\ \mathrm{Tr}_{B'}\, J_\Pi = \mathbb{1}_B/d_B \otimes \mathrm{Tr}_{BB'}\, J_\Pi$$

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- [Watrous, 2009] $\frac{1}{2}\|\mathcal{N}_1 - \mathcal{N}_2\|_\diamond$ given by SDP

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- Symmetry of id_m: Choi matrix invariant under $U \otimes \overline{U}$.

The one-shot ε-error quantum simulation cost $S^{(1)}_{\mathrm{NS},\varepsilon}(\mathcal{N})$

$$\frac{1}{2}\log \quad \text{minimize} \quad \operatorname{Tr} V_B \tag{8a}$$
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It reduces to the zero-error case when $\varepsilon = 0$ [Duan and Winter, 2016],

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What about $\varepsilon > 0$? Do we have

$$\frac{1}{2} I(A:B)_{\mathcal{N}} = S_{\mathrm{NS}}(\mathcal{N}) \xleftarrow{\text{i.i.d.}} S^{(1)}_{\mathrm{NS},\varepsilon}(\mathcal{N}) \overset{?}{=\!=} \frac{1}{2} H^{\varepsilon}_{\min}(A|B)_{J_{\mathcal{N}}} \xrightarrow{\text{i.i.d.}} \frac{1}{2} H(A|B)_{J_{\mathcal{N}}}$$

We will need a entropy that looks more similar to $I(A:B)_{\mathcal{N}}$.

The quantum mutual information of a state

$$I(A:B)_\rho := \inf_{\sigma_B} D(\rho_{AB}\|\rho_A \otimes \sigma_B). \tag{10}$$

The max-information of a quantum state [Berta et al., 2011]:

$$I_{\max}(A:B)_\rho := \inf_{\sigma_B} D_{\max}(\rho_{AB}\|\rho_A \otimes \sigma_B), \tag{11}$$

where max-relative entropy [Datta 2009] $D_{\max}(\rho\|\sigma) = \inf\{t|\rho \leq 2^t\sigma\}$.

Other variations can be seen in [Ciganović et al., 2013].

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$$I^{\varepsilon}_{\max}(A:B)_\rho := \inf_{\widetilde{\rho} \approx^{\varepsilon} \rho} I_{\max}(A:B)_{\widetilde{\rho}}. \tag{12}$$

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- We will generalize these notations and results to a **channel's version** and find their connection with the quantum channel simulation task.

Definition

For any quantum channel $\mathcal{N}_{A'\to B}$ we define the max-information of the channel $\mathcal{N}$ as

$$I_{\max}(A:B)_{\mathcal{N}} := I_{\max}(A:B)_{\mathcal{N}_{A'\to B}(\Phi_{AA'})}, \tag{14}$$

where $\Phi_{AA'}$ is the maximally entangled state.

We can replace $\Phi_{AA'}$ to any pure state $\phi_{AA'}$ with Schmidt rank $|A'|$.

Definition

The channel's smooth max-information is defined by

$$I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}} := \inf_{\substack{\frac{1}{2}\|\widetilde{\mathcal{N}}-\mathcal{N}\|_{\diamond}\leq\varepsilon \\ \widetilde{\mathcal{N}}\in\mathrm{CPTP}(A':B)}} I_{\max}(A:B)_{\widetilde{\mathcal{N}}}, \tag{15}$$

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Monotone under composition with CPTP maps, i.e., for any CPTP maps $\mathcal{N}_{A_1'\to B_1}$, $\mathcal{F}_{A_0'\to A_1'}$ and $\mathcal{T}_{B_1\to B_0}$,

$$I^{\varepsilon}_{\max}(A_0:B_0)_{\mathcal{T}\circ\mathcal{N}\circ\mathcal{F}} \le I^{\varepsilon}_{\max}(A_1:B_1)_{\mathcal{N}}. \tag{16}$$

$$D_{\max}(\mathcal{N}\|\mathcal{M}) := D_{\max}(J_{\mathcal{N}}\|J_{\mathcal{M}}) \tag{17}$$

$$I_{\max}(A:B)_{\mathcal{N}} = \inf_{\mathcal{M}\in\mathcal{G}} D_{\max}(\mathcal{N}\|\mathcal{M}) \tag{18}$$

$\mathcal{G} := \{\, \mathcal{M} \in \text{CPTP} : \exists\, \sigma \text{ s.t. } \mathcal{M}(\rho) = \sigma, \forall \rho \,\}$ the set of constant channels.

Resource theory perspectives

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Max-relative entropy and robustness for states: [Datta, 2009]

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Max-relative entropy and robustness for states: [Datta, 2009]

The robustness of a quantum channel $\mathcal{N}$ (see also [Díaz et.al, 2018])

$$\mathcal{R}_g(\mathcal{N}) := \inf\left\{t \geq 0 \,\middle|\, \exists\, \mathcal{M}\in \text{CPTP}(A:B) \text{ s.t. } \frac{\mathcal{N}+t\mathcal{M}}{1+t} \in \mathcal{G}\right\}. \tag{19}$$

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$$I_{\max}(A:B)_{\mathcal{N}} = \log[1 + \mathcal{R}_g(\mathcal{N})]. \tag{20}$$

Theorem

For any quantum channel $\mathcal{N}_{A'\to B}$ and given error tolerance $\varepsilon \geq 0$, we have

$$S^{(1)}_{\mathrm{NS},\varepsilon}(\mathcal{N}) = \frac{1}{2} I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}}. \tag{21}$$

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The AEP of the channel's smooth max-information ,

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n} I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} = I(A:B)_{\mathcal{N}}. \tag{22}$$

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Sketch of proof:

$$\begin{aligned}
\lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n} I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} &= 2\cdot \lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n} S^{(1)}_{\mathrm{NS},\varepsilon}(\mathcal{N}^{\otimes n}) \\
&= 2\cdot S_{\mathrm{NS}}(\mathcal{N}) && \text{[by definition]} \\
&= 2\cdot Q_E(\mathcal{N}) && \text{[Bennett et al., 2014]} \\
&= I(A:B)_{\mathcal{N}} && \text{[Bennett et al., 2002]}
\end{aligned}$$

Theorem

For any quantum channel $\mathcal{N}_{A' \to B}$ and given error tolerance $\varepsilon \geq 0$, we have

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Note: on the other hand, if we can proof Eq. (22) directly, it implies

$$Q_{\mathrm{E}}(\mathcal{N}) = Q_{\mathrm{NS}}(\mathcal{N}) = S_{\mathrm{NS}}(\mathcal{N}) \leq S_{\mathrm{E}}(\mathcal{N})$$

QRST

AEP proof outline

Direct proof of AEP "≥"

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} I_{\max}^{\varepsilon}(A:B)_{\mathcal{N}^{\otimes n}} \quad \geq \quad I(A:B)_{\mathcal{N}}$$

Direct proof of AEP "≥"

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} \quad \geq \quad I(A:B)_{\mathcal{N}}$$

$$I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} = \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n-\mathcal{N}^{\otimes n}\|_\diamond\leq\varepsilon} I_{\max}(A:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\phi^{\otimes n}_{AA'})} \qquad \text{[definition]}$$

$$= nI(A:B)_{\mathcal{N}_{A'\to B}(\phi_{AA'})} \qquad \text{[additivity]}$$

$$= nI(A:B)_{\mathcal{N}} \qquad [\text{optimal } \phi_{AA'}]$$

Direct proof of AEP "≥"

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} I_{\max}^{\varepsilon}(A:B)_{\mathcal{N}^{\otimes n}} \quad \geq \quad I(A:B)_{\mathcal{N}}$$

$$I_{\max}^{\varepsilon}(A:B)_{\mathcal{N}^{\otimes n}} = \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n - \mathcal{N}^{\otimes n}\|_\diamond \leq \varepsilon} I_{\max}(A:B)_{\widetilde{\mathcal{N}}^n_{A' \to B}(\phi^{\otimes n}_{AA'})} \qquad \text{[definition]}$$

$$\overset{[1]}{\geq} \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n - \mathcal{N}^{\otimes n}\|_\diamond \leq \varepsilon} I(A:B)_{\widetilde{\mathcal{N}}^n_{A' \to B}(\phi^{\otimes n}_{AA'})} \qquad [D_{\max} \geq D]$$

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[1] N. Datta, "Min-and max-relative entropies and a new entanglement monotone", 2009

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$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n} I_{\max}^{\varepsilon}(A:B)_{\mathcal{N}^{\otimes n}} \quad \geq \quad I(A:B)_{\mathcal{N}}$$

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\begin{aligned}
I_{\max}^{\varepsilon}(A:B)_{\mathcal{N}^{\otimes n}} &= \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n - \mathcal{N}^{\otimes n}\|_\diamond \leq \varepsilon} I_{\max}(A:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\phi^{\otimes n}_{AA'})} && \text{[definition]}\\
&\overset{[1]}{\geq} \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n - \mathcal{N}^{\otimes n}\|_\diamond \leq \varepsilon} I(A:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\phi^{\otimes n}_{AA'})} && [D_{\max} \geq D]\\
&\overset{[2]}{\gtrsim} I(A:B)_{\mathcal{N}^{\otimes n}_{A'\to B}(\phi^{\otimes n}_{AA'})} && \text{[continuity]}\\
&= nI(A:B)_{\mathcal{N}_{A'\to B}(\phi_{AA'})} && \text{[additivity]}\\
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\end{aligned}
$$

[1] N. Datta, "Min-and max-relative entropies and a new entanglement monotone", 2009

[2] R. Alicki, M. Fannes, "Continuity of quantum conditional information", 2004

Post-selection technique

$$\left\| \widetilde{\mathcal{N}}^n - \mathcal{N}^{\otimes n} \right\|_\diamond = \sup_{\phi^n_{AA'}} \left\| \widetilde{\mathcal{N}}^n_{A'\to B}(\phi^n_{AA'}) - \mathcal{N}^{\otimes n}_{A'\to B}(\phi^n_{AA'}) \right\|_1 \leq \varepsilon$$

$$\left\|\widetilde{\mathcal{N}}^n - \mathcal{N}^{\otimes n}\right\|_\diamond = \sup_{\phi^n_{AA'}} \left\|\widetilde{\mathcal{N}}^n_{A'\to B}(\phi^n_{AA'}) - \mathcal{N}^{\otimes n}_{A'\to B}(\phi^n_{AA'})\right\|_1 \leq \varepsilon$$

$\Uparrow$ [Christandl, König, Renner 2009]

$$\left\|\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{RAA'}) - \mathcal{N}^{\otimes n}_{A'\to B}(\omega^n_{RAA'})\right\|_1 \leq \varepsilon(n+1)^{-(|A'|^2-1)}$$

- $\omega^n_{RAA'}$ is a purification of the **de Finetti state**

$$\omega^n_{AA'} := \int \phi^{\otimes n}_{AA'} d(\phi_{AA'}),$$

  $d(\cdot)$ measure on normalized pure states induced by Haar measure;
- We can make $|R| \leq (n+1)^{|A'|^2-1}$.
  (see e.g. [Berta, Christandl, Renner 2011])

$$\left\|\widetilde{\mathcal{N}}^n - \mathcal{N}^{\otimes n}\right\|_\diamond = \sup_{\phi^n_{AA'}} \left\|\widetilde{\mathcal{N}}^n_{A'\to B}(\phi^n_{AA'}) - \mathcal{N}^{\otimes n}_{A'\to B}(\phi^n_{AA'})\right\|_1 \leq \varepsilon$$

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  (see e.g. [Berta, Christandl, Renner 2011])

We are ready to prove

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n} I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} \quad \leq \quad I(A:B)_{\mathcal{N}}$$

Direct proof of AEP "≤"

$$I_{\max}^{\varepsilon}(A:B)_{\mathcal{N}^{\otimes n}} = \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n - \mathcal{N}^{\otimes n}\|_{\diamond}\leq\varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{A'AR})} \qquad \text{[definition]}$$

Direct proof of AEP "≤"

$$I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} = \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n - \mathcal{N}^{\otimes n}\|_{\diamond} \leq \varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{A'AR})} \quad \text{[definition]}$$

$$\lesssim \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n(\omega^n_{A'AR}) - \mathcal{N}^{\otimes n}(\omega^n_{A'AR})\|_1 \leq \varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{A'AR})} \quad \text{[post-selection]}$$

Direct proof of AEP "≤"

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I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} &= \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n - \mathcal{N}^{\otimes n}\|_{\diamond}\le\varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{A'AR})} && \text{[definition]}\\
&\lesssim \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n(\omega^n_{A'AR}) - \mathcal{N}^{\otimes n}(\omega^n_{A'AR})\|_1\le\varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{A'AR})} && \text{[post-selection]}\\
&= \inf_{\substack{\frac{1}{2}\|\sigma^n_{BAR} - \mathcal{N}^{\otimes n}(\omega^n_{A'AR})\|_1\le\varepsilon \\ \sigma^n_{AR}=\omega^n_{AR}}} I_{\max}(AR:B)_{\sigma^n_{BAR}} && \text{[partial smooth]}
\end{aligned}
$$

Direct proof of AEP "≤"

$$I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} = \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n - \mathcal{N}^{\otimes n}\|_\diamond \le \varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{A'AR})} \quad \text{[definition]}$$

$$\lesssim \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n(\omega^n_{A'AR}) - \mathcal{N}^{\otimes n}(\omega^n_{A'AR})\|_1 \le \varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{A'AR})} \quad \text{[post-selection]}$$

$$= \inf_{\substack{\frac{1}{2}\|\sigma^n_{BAR} - \mathcal{N}^{\otimes n}(\omega^n_{A'AR})\|_1 \le \varepsilon \\ \sigma^n_{AR} = \omega^n_{AR}}} I_{\max}(AR:B)_{\sigma^n_{BAR}} \quad \text{[partial smooth]}$$

$$\overset{[1]}{\lesssim} I^{\varepsilon}_{\max}(AR:B)_{\mathcal{N}^{\otimes n}_{A'\to B}(\omega^n_{A'AR})} \quad \text{[global smooth]}$$

[1] A. Anshu, M. Berta, R. Jain, and M. Tomamichel, arXiv:1807.05630. Talk on Thursday by Mario.

Direct proof of AEP "≤"

$$I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} = \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n - \mathcal{N}^{\otimes n}\|_\diamond \leq \varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{A'AR})} \quad \text{[definition]}$$

$$\lesssim \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n(\omega^n_{A'AR}) - \mathcal{N}^{\otimes n}(\omega^n_{A'AR})\|_1 \leq \varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{A'AR})} \quad \text{[post-selection]}$$

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$$\overset{[2]}{\lesssim} I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}_{A'\to B}(\omega^n_{A'A})} \qquad \omega^n_{AA'} := \int \phi^{\otimes n}_{AA'} d(\phi_{AA'}) \quad \text{[get rid of } R\text{]}$$

[1] A. Anshu, M. Berta, R. Jain, and M. Tomamichel, arXiv:1807.05630. Talk on Thursday by Mario.

[2] M. Berta, M. Christandl, and R. Renner, "The quantum reverse Shannon theorem based on one-shot information theory", 2011.

Direct proof of AEP "≤"

$$I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} = \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^{n}-\mathcal{N}^{\otimes n}\|_{\diamond}\leq\varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^{n}_{A'\to B}(\omega^{n}_{A'AR})} \quad \text{[definition]}$$

$$\lesssim \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^{n}(\omega^{n}_{A'AR})-\mathcal{N}^{\otimes n}(\omega^{n}_{A'AR})\|_{1}\leq\varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^{n}_{A'\to B}(\omega^{n}_{A'AR})} \quad \text{[post-selection]}$$

$$= \inf_{\substack{\frac{1}{2}\|\sigma^{n}_{BAR}-\mathcal{N}^{\otimes n}(\omega^{n}_{A'AR})\|_{1}\leq\varepsilon \\ \sigma^{n}_{AR}=\omega^{n}_{AR}}} I_{\max}(AR:B)_{\sigma^{n}_{BAR}} \quad \text{[partial smooth]}$$

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$$\overset{[2]}{\lesssim} \max_{\phi_{AA'}} I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}_{A'\to B}(\phi^{\otimes n}_{AA'})} \quad \text{[quasi-convexity]}$$

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Direct proof of AEP "≤"

$$I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} = \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n - \mathcal{N}^{\otimes n}\|_{\diamond}\leq\varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{A'AR})} \quad \text{[definition]}$$

$$\lesssim \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n(\omega^n_{A'AR}) - \mathcal{N}^{\otimes n}(\omega^n_{A'AR})\|_1\leq\varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{A'AR})} \quad \text{[post-selection]}$$

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$$\overset{[2]}{\lesssim} \max_{\phi_{AA'}} nI(A:B)_{\mathcal{N}_{A'\to B}(\phi_{AA'})} \quad \text{[AEP for states]}$$

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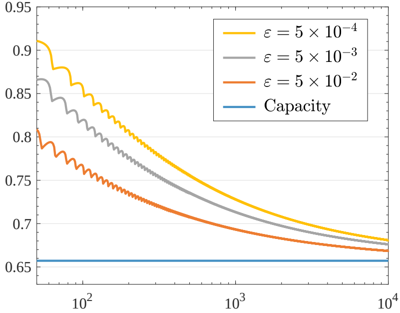
I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} &= \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n - \mathcal{N}^{\otimes n}\|_{\diamond} \leq \varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{A'AR})} && \text{[definition]} \\
&\lesssim \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n(\omega^n_{A'AR}) - \mathcal{N}^{\otimes n}(\omega^n_{A'AR})\|_1 \leq \varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{A'AR})} && \text{[post-selection]} \\
&= \inf_{\substack{\frac{1}{2}\|\sigma^n_{BAR} - \mathcal{N}^{\otimes n}(\omega^n_{A'AR})\|_1 \leq \varepsilon \\ \sigma^n_{AR} = \omega^n_{AR}}} I_{\max}(AR:B)_{\sigma^n_{BAR}} && \text{[partial smooth]} \\
&\overset{[1]}{\lesssim} I^{\varepsilon}_{\max}(AR:B)_{\mathcal{N}^{\otimes n}_{A'\to B}(\omega^n_{A'AR})} && \text{[global smooth]} \\
&\overset{[2]}{\lesssim} I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}_{A'\to B}(\omega^n_{A'A})} \qquad \omega^n_{AA'} := \int \phi^{\otimes n}_{AA'} d(\phi_{AA'}) && \text{[get rid of } R\text{]} \\
&\overset{[2]}{\lesssim} \max_{\phi_{AA'}} I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}_{A'\to B}(\phi^{\otimes n}_{AA'})} && \text{[quasi-convexity]} \\
&\overset{[2]}{\lesssim} \max_{\phi_{AA'}} nI(A:B)_{\mathcal{N}_{A'\to B}(\phi_{AA'})} && \text{[AEP for states]} \\
&= nI(A:B)_{\mathcal{N}} && \text{[definition]}
\end{aligned}
$$

[1] A. Anshu, M. Berta, R. Jain, and M. Tomamichel, arXiv:1807.05630. Talk on Thursday by Mario.

[2] M. Berta, M. Christandl, and R. Renner, "The quantum reverse Shannon theorem based on one-shot information theory", 2011.

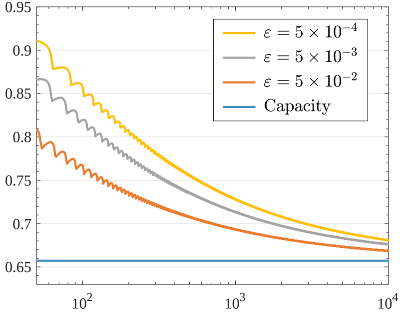
Depolarizing channel: $\mathcal{N}(\rho) = (1-p)\rho + p \cdot \mathbb{1}/d, \qquad p = 0.15.$



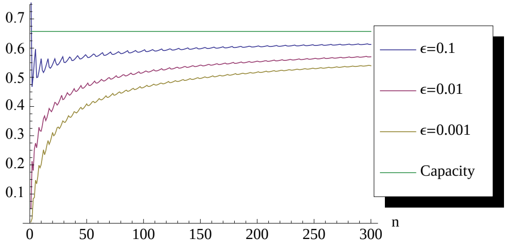
$$
\begin{aligned}
S^{(1)}_{\mathrm{NS},\varepsilon}(\mathcal{N}) &= \frac{1}{2} I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}} \\
&= \frac{1}{2} \min_{\mathcal{M} \in \mathcal{G}} D^{\varepsilon}_{\max}(\mathcal{N} \| \mathcal{M}) \\
&\to \frac{1}{2} I(A:B)_{\mathcal{N}}.
\end{aligned}
$$

Example

Depolarizing channel: $\mathcal{N}(\rho) = (1-p)\rho + p \cdot \mathbb{1}/d, \qquad p = 0.15.$



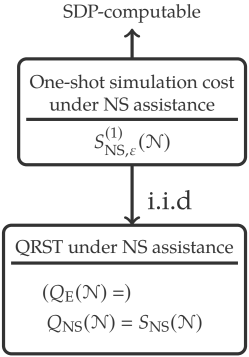
$$S^{(1)}_{\mathrm{NS},\varepsilon}(\mathcal{N}) = \frac{1}{2} I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}}$$
$$= \frac{1}{2} \min_{\mathcal{M}\in\mathcal{G}} D^{\varepsilon}_{\max}(\mathcal{N}\|\mathcal{M})$$
$$\to \frac{1}{2} I(A:B)_{\mathcal{N}}.$$



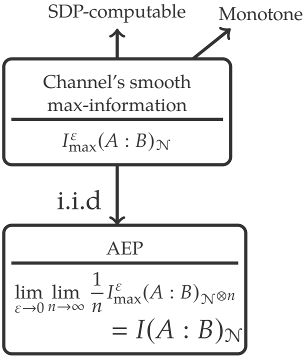
Matthews-Werner converse bound 2014

$$MW(\mathcal{N},\varepsilon) = \frac{1}{2} \min_{\mathcal{M}\in\mathcal{G}} D^{\varepsilon}_{H}(\mathcal{N}\|\mathcal{M}) \to \frac{1}{2} I(A:B)_{\mathcal{N}}$$

Channel simulation task



Channel's max-information



Discussions

- How to better characterize the one-shot entanglement-assisted channel simulation cost?

- How to better characterize the one-shot entanglement-assisted channel simulation cost?
- More general form of channel AEP?
  (presented by Andreas in JILA workshop open problem session)
  Channel divergence:

$$\boldsymbol{D}(\mathcal{N}\|\mathcal{M}) := \max_{\phi_{AA'}} \boldsymbol{D}(\mathcal{N}_{A'\to B}(\phi_{AA'})\|\mathcal{M}_{A'\to B}(\phi_{AA'})). \qquad (23)$$

What we already know for $\mathcal{G}_n$ the set of constant channels:

$$\min_{\mathcal{M}^n\in\mathcal{G}_n} D_{\max}^{\varepsilon}(\mathcal{N}^{\otimes n}\|\mathcal{M}^n) \xrightarrow{\text{this work}} \min_{\mathcal{M}\in\mathcal{G}} D(\mathcal{N}\|\mathcal{M}) \quad \checkmark \qquad (24)$$

$$\min_{\mathcal{M}^n\in\mathcal{G}_n} D_{H}^{\varepsilon}(\mathcal{N}^{\otimes n}\|\mathcal{M}^n) \xrightarrow{\text{MW bound}} \min_{\mathcal{M}\in\mathcal{G}} D(\mathcal{N}\|\mathcal{M}) \quad \checkmark \qquad (25)$$

What if $\mathcal{G}_n = \{\mathcal{M}^{\otimes n}\}$? Quantum channel Stein's lemma?

Thanks for your attention!

See arXiv:1807.05354

for more details